LOG MINIMAL MODELS FOR ARITHMETIC THREEFOLDS

by

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STATEMENT OF DISSERTATION APPROVAL

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ABSTRACT

I study the existence of log minimal models for a Kawamata log-terminal pair of relative dimension two over a Dedekind domain. This generalizes the semistable result of Kawamata [Kaw94, Kaw99]. Also I prove a result on the invariance of log plurigenera for such pairs, generalizing the result of [Suh08]. To extend the result from discrete valuation rings to Dedekind domains, some computability results are given for basepoint-freeness, vanishing of cohomology, and finite generation of log-canonical and adjoint rings on a mixed characteristic family of surfaces.

To my wife Lifang and my dog Pigu.

CONTENTS

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CHAPTER 1

INTRODUCTION

Algebraic Geometry deals with the study of geometric objects defined by zero sets of collections of polynomials. For example, the equation $0 = v^2 - u^3 - 486662u^2 - u$ defines a one dimensional geometric object called an elliptic curve. The Minimal Model program is an attempt to classify all such objects into birational equivalence classes, where two such objects are birationally equivalent if there is some map, for example a substitution of variables, which transforms one into the other. For example, the above curve equation can be transformed to the equation $0 = x^2 + y^2 - 1 - \frac{121665}{121666}x^2y^2$ by setting:

$$v = \sqrt{486662}\frac{u}{x}$$

$$u = \frac{1+y}{1-y}.$$

It is unknown in general, even over very simple fields such as the complex numbers, whether every variety has a minimal model. However, recently there have been some important results in this area, such as the proof by Birkar, Cascini, Hacon, and Mckernan, that every complex variety of "general type" (i.e. the vast majority of such varieties, but excluding many special cases) has a minimal model [BCHM10].

Arithmetic geometry is the study of objects, such as the above curves, over more general domains, such as the integers. These objects can be thought of as existing in a family, since the integers themselves (in algebraic geometry terms) have dimension 1. Alternatively, since curves have dimension 1, and the integers have dimension 1, such an object is called an "arithmetic surface." The existence of minimal models in the birational classification of arithmetic surfaces is known due to the following result:

Theorem 1.1. *[Liu02, 9.3.19] Let $f : X \to S$ be an arithmetic surface. Then there exists a birational morphism $X \to Y$ of arithmetic surfaces over S, with Y relatively minimal.*

In this thesis, I increase the dimension by one, and ask whether such a result holds for arithmetic threefolds. After a brief introductory chapter discussing general aspects of the Minimal Model Program, I specifically study the existence of minimal models in the birational classification of pairs of objects, having dimension two over a one dimensional arithmetic domain such as the integers. The result is an affirmative answer, with some mild conditions on how singular the space is, that an arithmetic threefold does have a minimal model.

I begin to prove some of the main results in this thesis in chapter 3, by studying the invariance of higher log plurigenera for a family of surfaces over a discrete valuation ring having mixed characteristic. That chapter begins with a brief recollection of some previous results due to Katsura and Ueno, Suh, and Tanaka. I then state several necessary preliminary and known facts which I use to prove the following two theorems:

Theorem A . *(Invariance of Plurigenera: Theorem 3.23) Let* (X, Δ) *be a Kawamata log-terminal pair of relative dimension* 2 *over a discrete valuation ring* R *with perfect residue field* k *of characteristic* $p > 0$ *and perfect fraction field* K*. Assume that* $K_X + \Delta$ *is, big,* $\mathbb{Q}$*-Cartier, and simple normal crossings over* R*. Then, there exists an* m_0 *depending on the intersection numbers, such that for* $m \in m_0\mathbb{Z}^+$ *,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Theorem B . *(Invariance of Kodaira Dimensions: Theorem 3.24) Let* (X, Δ) *be a Kawamata log-terminal pair of relative dimension* 2 *over a discrete valuation ring* R *with residue field* k *and fraction field* K *such that* k *is algebraically closed of characteristic* $p > 0$*. Assume* $K_X + \Delta$ *is pseudo-effective and simple normal crossings over* R*. Then the numerical Kodaira dimensions satisfy*

$$\nu\left(K_{X_K} + \Delta_K\right) = \nu\left(K_{X_k} + \Delta_k\right).$$

As a result of abundance for log surfaces, the log Kodaira dimensions also satisfy

$$\kappa\left(K_{X_K} + \Delta_K\right) = \kappa\left(K_{X_k} + \Delta_k\right).$$

In Chapter 4, I use Theorem 3.23 to prove the Abundance Theorem for general type arithmetic threefolds over a Dedekind domain R with perfect residue and fraction fields.

Theorem C . *(Abundance: Theorem 4.10) Let* (X, Δ) *be a big, Kawamata log-terminal pair of relative dimension* 2 *over* R*. If* $K_X + \Delta$ *is nef, then it is semi-ample.*

I also prove in Chapter 4 various results related to finite generation of adjoint and canonical rings which end up extending to the whole family, e.g. to $\mathbb{Z}$ if the space is an arithmetic threefold. This extension relies on showing that a given family has a bound on the degrees of generation for the whole family. Thus I additionally prove some results related to vanishing of cohomology. The main finite generation theorem of chapter 4 is the following:

Theorem D . *(Finite Generation: Theorem 4.21) Let X/R be an arithmetic threefold. Let $\{(X, \Delta_i)\}_{i\in\{1,...,k\}}$ be a big, log-smooth, $\mathbb{Q}$-Cartier KLT pair for each i and such that $\sum_{i=1}^{k} \Delta_i$ has simple normal crossings support. Then there exists a constant m_0 such that on any fiber $X_r, r \in R$ the ring*

$$R\left(X_r, K_X + \Delta_1|_{X_r}, ..., K_X + \Delta_k|_{X_r}\right)$$

is finitely generated in degree m_0. Furthermore, the ring

$$R(X, K_X + \Delta_1, ..., K_X + \Delta_k)$$

is finitely generated.

In Chapter 5, I begin the study of the actual minimal model program for arithmetic threefolds over a Dedekind Domain R with perfect residue fields. The main ingredients of the usual Minimal Model program such as the Cone Theorem, the Rationality theorem, and the existence of flips, are proven in the arithmetic threefold setting, and the main result is the termination of the Minimal Model Program with scaling in the general type case:

Theorem E . *(Big Termination With Scaling: Theorem 5.10) Let (X, Δ) be a projective $\mathbb{Q}$-factorial Kawamata log-terminal pair of relative dimension 2 over R. Suppose $K_X + \Delta$ is big and log smooth over R. Then the minimal model program for (X, Δ) can be run, resulting in a terminating sequence of flips and divisorial contractions.*

Chapter 6 uses the general type termination with scaling from Chapter 5 to prove the existence, in the non-general type case of minimal models of arithmetic threefolds. The techniques are different because the lifting result, Theorem A, applies only in the general type case, and thus finite generation is not yet known (except maybe in some trivial cases, like over a field). Nevertheless, applying similar reductions to [HMX14], the following is deduced:

Theorem F . *(Existence of Minimal Models: Theorem 6.9) Let (X, Δ) be a Kawamata log-terminal pair of relative dimension 2, proper over R. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, pseudo-effective, and log smooth over R. Then the minimal model of (X, Δ) exists.*

The above result is somewhat related to a theorem of Kawamata [Kaw94, Kaw99] where he proves the terminal[1], semistable case of the above assuming no boundary and in characteristics $p \geq 5$. However, the techniques used in this paper are mostly quite different from Kawamata's paper, and the motivating terminal case for the above theorem would more accurately be the proof of the existence of minimal models over certain nice discrete valuation rings given by Katsura and Ueno [KU85]. Other somewhat similar results are the recent proofs of the existence of terminal and Kawamata log-terminal minimal models respectively in positive characteristic for threefolds over a field in [HX15, Bir15]. However, the case considered here is the case of geometric dimension 2 and arithmetic dimension 1, rather than geometric dimension 3 and arithmetic dimension 0. Another somewhat similar result in geometric dimension 3, arithmetic dimension 0 and characteristic $p > 5$ is [BCZ15, 1.6] where the existence of log minimal models over a curve in equal characteristic is proven over an algebraically closed field (note that the Theorem F can apply more generally in equal characteristic over a perfect field as well even without the restriction that $p > 5$.)

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[1] Note that the [Kaw99] redacts some of the statements from [Kaw94], in particular, the proof in characteristics 2 and 3.

CHAPTER 2

THE MINIMAL MODEL PROGRAM

The Minimal Model Program is an attempt to classify certain algebraically defined objects by putting them into equivalence classes which are reached after removing some non-crucial parts. In this chapter I discuss the basic aspects of the minimal model program and arithmetic threefolds.

2.1 Arithmetic Schemes

Here are the basic objects I will be concerned with:

Definition 2.1. A **perfect field** is a field which either has characteristic 0 or where every element is a pth power. A **discrete valuation ring** is a domain where unique factorization holds having a unique irreducible element. A **Dedekind domain** is a Noetherian domain such that the localization at each maximal ideal is a discrete valuation ring (for example all principal ideal domains, and the ring of integers in a number field are Dedekind domains). An **arithmetic scheme** is a scheme of finite type over $\mathbb{Z}$ or more generally over a characteristic zero Dedekind domain (I will assume all Dedekind domains mentioned in this paper are characteristic 0 domains such as $\mathbb{Z}$ while the residue fields can have any characteristic). An **arithmetic threefold** is a family of surfaces which is projective over a Dedekind domain.

For an example of an arithmetic threefold, see the compact Shimura varieties studied in [Suh08].

In the minimal model program we use maps from one scheme to another in order to reach the minimal model.

Definition 2.2. A **birational morphism** is a morphism of finite type $f : Z \to X$ which is a bijection on the set of generic points (i.e. $f^{-1}(\eta_i) = \eta_i'$) giving an isomorphism of the rings $\mathcal{O}_{X,\eta_i} \to \mathcal{O}_{Z,\eta_i'}$.

A special type of birational morphism is a "resolution of singularities," which can contract some codimension 1 subschemes (divisors) whose components intersect transversely according to the following definition:

Definition 2.3. [Kol13, 1.7] Let X be a scheme. Let $p \in X$ be a regular point with ideal sheaf $\mathfrak{m}_p$ and residue field $k(p)$. Then $x_1, ..., x_n \in \mathfrak{m}_p$ are called local coordinates if their residue classes $\overline{x_1}, ..., \overline{x_n}$ form a $k(p)$-basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$. Let $D = \sum a_i D_i$ be a Weil divisor on X. We say that (X, D) has **simple normal crossings** or snc at a point $p \in X$ if X is regular at p and there is an open neighborhood $p \in X_p \subset X$ with local coordinates $x_1, ..., x_n \in \mathfrak{m}_p$ such that $X_p \cap Supp\ D \subset (x_1 \cdot \cdots \cdot x_n = 0)$. Irreducible components of D and their intersections are called the **strata** of D. We say that (X, D) is snc if it is snc at every point. We say that (X, D) has **normal crossing** or nc at a point $p \in X$ if $\left(\hat{X}_K, D|_{\hat{X}_K}\right)$ is snc at p where $\hat{X}_K$ denotes the completion at p and K is a separable closure of $k(p)$. We say that (X, D) is nc if it is nc at every point. If (X, D) is defined over a perfect field, this concept is also called **log smooth**.

In the case of arithmetic threefolds, the resolution of singularities exists by the following theorem:

Theorem 2.4. *[CP09, 1.1] Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*

1. *X' is everywhere regular*
2. *π induces an isomorphism $\pi^{-1}(Reg\ X) \approx Reg\ X$*
3. *$\pi^{-1}(Sing\ X)$ is a strict normal crossings divisor on X'.*

Thus, in the Minimal Model Program it is natural to consider pairs of objects (X, Δ), with Δ a simple normal crossings divisor on X, called a "boundary divisor". Note that the divisor $\pi^{-1}(Sing\ X)$ produced by the above theorem is usually called "exceptional." In the case where a pair (X, Δ) is considered, the resolution of singularities is generalized to a log resolution, which is defined as follows:

Definition 2.5. A **log resolution** of a pair (X, Δ) is a proper birational morphism $f : Y \to X$ from a regular scheme such that the exceptional locus $Ex(f)$ is a divisor and $f^{-1}(\Delta) \cup Exc(f)$ has simple normal crossings support.

In the log smooth, arithmetic threefolds case, log resolutions exist by the following theorem:

Theorem 2.6. *[CP09, 4.3] Let S be a regular Noetherian irreducible scheme of dimension three which is excellent and $\mathcal{I} \subset \mathcal{O}_S$ be a nonzero ideal sheaf. There exists a finite sequence*

$$S := S(0) \leftarrow S(1) \leftarrow \cdots \leftarrow S(r)$$

with the following properties:

1. *For each j, $0 \leq j \leq r-1$, $S(j+1)$ is the blowing up along a regular integral subscheme $\mathcal{Y}(j) \subset \mathcal{S}(j)$ with*

$$\mathcal{Y}(j) \subseteq \left\{ s_j \in \mathcal{S}(j) : \mathcal{I}\mathcal{O}_{S(j),s_j} \text{ is not locally principal} \right\}.$$

2. *$\mathcal{I}\mathcal{O}_{S(r)}$ is locally principal.*

Recall that the "canonical divisor", K_X, of a given scheme or variety X is defined by taking the local equations given by tensoring together a basis for the sheaf of differentials on X. After taking a log resolution, we can classify how bad the singularities of the original scheme were according to the coefficients of the exceptional divisor of the pullback of the canonical:

Definition 2.7. Let X be a normal scheme and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier (locally defined by a single equation). Then the pair (X, Δ) has **terminal** (respectively **Kawamata log-terminal**, **log-canonical**) singularities if for any log resolution $f : Y \to X$ of (X, Δ) such that E_i are exceptional curves on Y, then

$$K_Y + \Delta_Y = f^*(K_X + \Delta) + \sum a_i E_i$$

where $a_j > 0$ (respectively $a_j > -1$, $a_j \geq -1$) and Δ_Y is the strict transform of Δ. The coefficients a_i here are called the **log discrepancies** of the divisors E_i. If there exists at least one log resolution such that all the $a_i > -1$, then (X, Δ) is said to be **divisorially log-terminal**.

2.2 Intersections and Positivity

Since we care about intersections among the components of Δ, the exceptional divisors, and various other divisors which will appear, we should know how these things intersect. The intuition is that on a surface over an algebraically closed field, divisors (which are curves in this case) should have an intersection number corresponding to the number of times they actually geometrically intersect. As noted in [KU85], for a projective surface S over a field, if D_1, D_2 are two divisors with corresponding invertible sheaves L_1, L_2, then this intersection number is given by the coefficient $n_1 n_2$ of $\chi(S, L_1^{n_1} \otimes L_2^{n_2})$ where χ is the Euler characteristic. Noting that the Euler-characteristic is invariant in a flat family [Har11, 8.4.6], we have the following (stated there when the residue field is algebraically closed, but holding in general):

Lemma 2.8. *[KU85, 9.3] Let $\varphi : X \to Spec(R)$ be a smooth, proper family of surfaces over a discrete valuation ring R. If D, D' are divisors on X, then*

$$\left(D_k \cdot D'_k\right)_{X_k} = \left(D_K \cdot D'_K\right)_{X_K}.$$

If X/R is smooth, then lemma 2.8 applies to show that for any prime divisors C, D extending to both fibers, we can just define $C.D = (C|_{X_k}.D|_{X_k})_{X_k}$. On the other hand, if Y/R is merely normal and proper, but is actually a scheme, the resolution of singularities, theorem 2.4, holds. Thus the intersection theory can be defined as in [Tan14, Def 3.1]: $f : X' \to X$ is a resolution, and $C.D = f^*C.f^*D$ for two divisors C, D on Y/R. By properness (and since the fibers are two-dimensional), this intersection extends by linearity to Weil divisors with $\mathbb{Q}$ or $\mathbb{R}$ coefficients. Numerical equivalence and $N^1(X)_{\mathbb{Q},\mathbb{R}}$ are then defined as usual.

Using the above definition, it is possible to achieve certain intersection numbers which are unintuitive. For example, consider a map $f : Y \to X$ which is a birational morphism contracting a curve E to a point x. If H, D are two divisors on X which intersect at one point with multiplicity 1, then pulling them back to Y will result in different intersections depending on whether their intersection point is x or not. If the point is not x, then their intersection on Y will not change. However, if they intersect at x, then on Y, their intersection will consist of their two intersections with E:

$$D.H = (D + E).(H + E) = D.E + E.H + E^2$$

Thus it must be the case that E^2 is negative. In the case of smooth complex surfaces, the minimal model program tries to contract all curves (i.e. one-dimensional subschemes on X) which have self-intersection -1.

The following definition concerns the positivity of intersections of a given divisor.

Definition 2.9. A divisor D on a scheme X is called **ample** (respectively **nef**) if it intersects positively (respectively non-negatively) with every positive-dimensional irreducible subvariety $V \subset X$. D is called **big** if it has positive self-intersection.

A divisor being "big" is equivalent to it having Kodaira dimension equal to the dimension of the underlying space:

Definition 2.10. The (log) **Kodaira dimension** $\kappa(K_X + \Delta)$ of a pair (X, Δ) over an algebraically closed field k is the transcendence degree, over k, of the **log canonical ring**

$$\mathfrak{R} = \bigoplus_{n \geq 0} H^0(X, n(K_X + \Delta))$$

We can also define the **numerical Kodaira dimension** $\nu(K_X + \Delta)$ in the same manner via the following ring:

$$\mathfrak{R}' = \bigoplus_{n \geq 0} H^0(X, n(K_X + \Delta) + H)$$

for a fixed ample divisor H. A pair is called **abundant** if these dimensions are equal.

Note that, on an arithmetic scheme X, the nef and ample properties can be decided by looking at the restriction to the fibers X_s:

Theorem 2.11. *Let S be an Dedekind domain and $f : X \to S$ a projective morphism. Let D be divisor on X such that D_s is nef (respectively ample) for every closed point $s \in S$. Then D is nef (respectively ample).*

Proof. The ample case is [Liu02, 5.3.24]. Thus assume that D_s is nef at all closed points. Now, c.f. [Laz04, 1.4.10], it suffices to choose an ample divisor H which restricts to X_s, so that $D_s + \epsilon H_s$ is ample for all sufficiently small ϵ. Then $D + \epsilon H$ is ample for all sufficiently small ϵ, and so D is nef. □

The general technique for deciding which subschemes should be contracted to reach a minimal model is to use the "Cone Theorem" which separates the cone of all curves on X, $NE(X)$, into those intersecting positively and non-positively with the canonical divisor K_X. For surfaces over an algebraically closed field in positive characterstic, this is due to [KK94], although I use the updated version given recently by Tanaka [1]

Theorem 2.12. *[Tan14, 4.4] Let X be a projective normal surface and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let H be an $\mathbb{R}$-Cartier ample $\mathbb{R}$-divisor. Then the following assertions hold:*

1. $\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$

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[1] Tanaka notes that a version holds over non-closed fields in [Tan15a, 1.17]

2. $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i]$

3. Each C_i *in (1) and (2) is rational or* $C_i = B_j$ *for some* B_j *with* $B_j^2 < 0$*.*

On a surface, the log minimal model program proceeds by taking a sequence of birational morphisms $f_1 : (X, \Delta) \to (X_2, \Delta_2)$, $f_2 : (X_2, \Delta_2) \to (X_3, \Delta_3)$, ... contracting the curves C_i given in the above theorem, until a final "minimal" model X_n is reached with $K_{X_n} + \Delta_n$ nef. In other words, the log minimal model is the (X_n, Δ_n) such that

$$\overline{NE}(X_n)_{K_{X_n}+\Delta_n\geq 0} = \overline{NE}(X_n).$$

Note that in the second part of the cone theorem, the cone of curves becomes simpler when an ample divisor H is added to the pair $K_X + \Delta$. For this reason we define a special type of minimal model program, called the minimal model program with scaling.

Definition 2.13. [HK00a, 5.E] If $K_X + \Delta$ is an effective Kawamata log-terminal pair on a two-dimensional variety X, then for any ample divisor H', we can find $h \in \mathbb{R}_{>0}$ and an $\mathbb{R}$-divisor $H \sim_{\mathbb{R}} hH'$ such that $(X, \Delta + H)$ is Kawamata log-terminal and $K_X + \Delta + H$ is nef and big. Let

$$\lambda = inf\{t \geq 0 \mid K_X + \Delta + tH \ is \ nef\}.$$

(This is called the **nef threshold** of $K_X+\Delta$.) If $\lambda = 0$, then $K_X+\Delta$ is nef, and thus (X, Δ) is minimal. If $\lambda > 0$, then, by Theorem 2.12, there exists a $(K_X + \Delta)$-negative extremal ray R such that $(K_X + \Delta + \lambda H).R = 0$. Then, c.f. [KK94, 2.3], if the corresponding contraction $\phi : X \to X'$ does not result in a log Del Pezzo surface or a birationally ruled surface, then setting $H' = \phi_* H$ and $\Delta' = \phi_*\Delta$, the divisor $K_{X'} + \Delta' + \delta H'$ is nef. Then the process may be repeated. The process either terminates at some step in a log minimal model or at one of the aforementioned surfaces. The end result is a finite sequence of real numbers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ such that $K_{X_n} + \Delta_n + \lambda_n H_n$ is nef and $X \to X_n$ is a minimal model for $(X, \Delta + \lambda_n H)$.

In fact, if $K_X + \Delta$ is pseudo-effective (the limit of divisors defined locally by actual equations), then adding an ample H to it will result in a big divisor. Another reason to use the minimal model program with scaling is that divisors which are both big and nef sometimes satisfy nice cohomological properties, such as the vanishing of their higher cohomology groups. Arguments based in cohomology are sometimes preferrable, since cohomology acts nicely under base change. The vanishing result which seems most useful in the arithmetic threefold situation is the following result due to Tanaka:

Theorem 2.14. *(X-method Vanishing [Tan15c, 2.11]) Let* (X, Δ) *be a projective Kawamata log-terminal surface (over an algebraically closed field of characteristic* $p > 0$*) where* Δ *is an effective* $\mathbb{R}$*-divisor. Let* N *be a nef* $\mathbb{R}$*-cartier* $\mathbb{R}$*-divisor with* $\nu(X, N) \geq 1$*. Let* D *be a* $\mathbb{Q}$*-Cartier* $\mathbb{Z}$*-divisor such that* $D - (K_X - \Delta)$ *is nef and big. Then there exists a positive real number* $r(\Delta, D, N)$ *such that*

$$H^i\left(X, \mathcal{O}_X\left(D + rN + N'\right)\right) = 0$$

for every $i > 0$*, every positive real number* $r \geq r(\Delta, D, N)$*, and every nef* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* N' *such that* $rN + N'$ *is a Cartier divisor.*

2.3 Contractions and Flips

In the surface case, it is possible to contract codimension one subschemes, but in higher dimensions, contractible two-dimensional subschemes exist.

Definition 2.15. Let $f : X \to Z$ be a projective birational morphism of algebraic spaces such that $f_*\mathcal{O}_X = \mathcal{O}_Z$ and *dim* $NE(X/Z) = 1$ and f contracts some divisor. Then f is called a **divisorial contraction**. If instead f contracts some subvariety of codimension ≥ 2 and no divisors, then f is called a **small contraction**.

When running the minimal model program with scaling, as in Definition 2.13, the contraction at step i is given by taking Proj of the log-canonical ring of $K_{X_i}+\Delta_i+t_iH_i$. One must therefore prove that this ring is finitely generated. For surfaces there is the following:

Theorem 2.16. *[Tan14, 7.1] Let* X *be a projective normal* $\mathbb{Q}$*-factorial surface over* k *and let* Δ *be a* $\mathbb{Q}$*-boundary. Then*

$$R(X, K_X + \Delta) := \bigoplus_{m\geq 0} H^0(X, \llcorner m(K_X + \Delta)\lrcorner)$$

is a finitely generated k*-algebra.*

One useful tool for studying finite generation of the above ring is the Invariance of Plurigenera theorem, which allows one to reduce finite generation of the log-canonical ring, to finite generation on a sub-scheme. In the arithmetic threefold case, only special cases of this result have been proven before this thesis, however, in characteristic 0, this famous result exists due to Siu:

Theorem 2.17. *[Siu98] Let $\pi : X \to T$ be a smooth projective family of compact complex manifolds parametrized by the open unit 1-disk T. Assume that the family $\pi : X \to T$ is of general type. Then for every positive integer m, the plurigenus*

$$P_m(X_t) := dim_{\mathbb{C}} H^0\left(X_t, mK_{X_t}\right)$$

is independent of $t \in T$, where $X_t = \pi^{-1}(t)$ and K_{X_t} is the canonical line bundle of X_t.

The above theorem in characteristic 0 has been generalized to all Kodaira dimensions and to Kawamata log-terminal, log smooth pairs c.f. [Siu02], [HMX13, 1.8], and [BP12].

In dimensions greater than two, taking Proj of the log-canonical ring will result in either a small or divisorial contraction. In the case of a small contraction $f : X \to Y$, the resulting space will have the property that no multiple of $K_X + \Delta$ is Cartier (this property is called $\mathbb{Q}$-factoriality, and it is implied by log-smoothness) for the following reason: $m(K_X + \Delta)$ and $f^*(m(K_Y + \Delta'))$ are linearly equivalent, since they agree outside of a codimension-two subset. However their intersections with the contracted extremal ray are not the same [KM98, 2.6].

Since many of cohomological techniques require a multiple of $K_X + \Delta$ to be Cartier, this situation is undesirable. The solution is to produce an additional space, which is $\mathbb{Q}$-factorial, on which the contracted ray remains contracted. This space is called the "flip:"

Definition 2.18. [KM98, 3.33] Let X be a $\mathbb{Q}$-factorial normal scheme and D a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. A **$(K_X + D)$-flipping contraction** is a proper birational morphism $f : X \to Y$ to a normal scheme Y such that $Exc\,(f)$ has codimension at least two in X and $-\,(K_X + D)$ is f-ample. A normal scheme X^+ together with a proper birational morphism $f^+ : X^+ \to Y$ is called a **$(K + D)$-flip** of f if

(1) $K_{X^+} + D^+$ is $\mathbb{Q}$-Cartier, where D^+ is the birational transform of D on X^+

(2) $K_{X^+} + D^+$ is f^+-ample, and

(3) $Ex\,(f^+)$ has codimension at least two in X^+.

The induced rational map $\phi : X \to X^+$ is sometimes called a $(K + D)$-flip by abuse of notation.

Yet another reason for studying the log-canonical ring is that existence of the flip can be reduced to showing that the log-canonical ring is finitely generated. After proving these objects exist, the final obstacle in the log-minimal model program is usually to show that there is no infinite sequence of flips, i.e. that flips terminate.

In characteristic 0, and in low-dimensions, many of the above results have been proven. For example, [BCHM10] proves termination in the general type case, while termination is proven for positive characteristic surfaces in [KK94].

CHAPTER 3

INVARIANCE OF LOG PLURIGENERA

The purpose of this chapter is to consider the mixed characteristic analogue of Theorem 2.17 in the case of an algebraic variety X/R of relative dimension 2 over a discrete valuation ring (and consequences).

3.1 Introduction

In positive characteristic, the invariance of plurigenera does not hold in general. When the fibers of a family of surfaces have Kodaira dimension 1 there are examples of Katsura and Ueno [KU85] where a fiber with wild ramification causes the geometric genus to jump rather than being constant in the family. Similarly, Suh [Suh08] has constructed counter-examples in Kodaira dimension 2 to the invariance of the geometric genus.

However, in their paper, Katsura and Ueno also show that in the case of a smooth algebraic variety X/R over a discrete valuation ring of relative dimension 2 with residue field k and fraction field K, then $\kappa\left(X_k\right) = \kappa\left(X_K\right)$. So the question becomes whether some asymptotic version of the invariance holds in this case. Specifically, the question this chapter seeks to answer is whether for $m \gg 0$, it holds that

$$P_m(mK_{X_k}) := dim_k H^0\left(X_k, mK_{X_k}\right) = dim_K H^0\left(X_K, mK_{X_K}\right) = P_m(mK_{X_K}),$$

and if more generally, the same holds for Kawamata log-terminal pairs. As far as the author is aware, the only existing results in this direction are the following result due to Junescue Suh, which uses the techniques of [KU85], as well as a W_2-lifting hypothesis in place of the Kawamata-Viehweg vanishing theorem for characteristic 0, and a result due to Tanaka, which assumes a certain ample divisor is added to the pair. Suh's theorem is:

Theorem 3.1. *[Suh08, 1.2.1(ii), 1.2.4] Let R be a discrete valuation ring whose fraction field K (resp. residue field k) has characteristic zero (resp. is perfect of characteristic $p > 0$) and let X/R be a proper smooth algebraic space of relative dimension 2. If X_k lifts to $W_2\left(k\right)$ and is of general type, then one has*

$$P_m\left(X_K\right) = P_m\left(X_k\right)$$

for every integer $m \geq 2$*. If moreover* X_k *has reduced picard scheme then* $P_m(X_K) = P_m(X_k)$ *for all* $m \geq 1$*.*

Tanaka's theorem is:

Theorem 3.2. *[Tan15b, 7.3] Let* X *be a smooth projective threefold. Let* S *be a smooth prime divisor on* X *and let* A *be an ample* $\mathbb{Z}$*-divisor on* X *such that*

1. $K_X + S + A$ *is nef, and*
2. $\kappa(S, K_S + A|_S) \neq 0$*.*

Then there exists $m_0 \in \mathbb{Z}_{>0}$ *such that, for every integer* $m \geq m_0$*, the natural restriction map*

$$H^0(X, m(K_X + S + A)) \to H^0(S, m(K_S + A|_S))$$

is surjective.

The proof of the above theorem uses some interesting trace of Frobenius methods which are quite different from the techniques used here. Instead, I use minimal model techniques, combined with the methods of [KU85], to gain a result similar to the above theorems but without the $W_2(k)$-lifting hypothesis or the ample $\mathbb{Z}$-divisor A, and with the added benefit that it holds for Kawamata log terminal pairs. The main theorem of the Chapter is the following:

Theorem 3.3. *Let* (X, Δ) *be a Kawamata log-terminal pair of relative dimension* 2 *over a discrete valuation ring* R *with perfect residue field* k *of characteristic* $p \geq 2$ *and perfect fraction field* K*. Assume that* (X, Δ) *is pseudo-effective,* $\mathbb{Q}$*-Cartier, and simple normal crossings over* R*. If* $\nu(K_X + \Delta) \neq 1$*, then there exists an* m_0 *such that for all* $m \in \mathbb{Z}^+$ *with* $m_0 | m$*,*

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

Consequences of the above theorem are the invariance of the log Kodaira dimensions of Kawamata log-terminal arithmetic threefold pairs, the finite generation of the log-canonical ring (which will be discussed in the next chapter), and the ability to run a log minimal model program in mixed characteristic and relative dimension 2 (discussed in chapters 5 and 6.)

Corollary 3.4. *Let* (X, Δ) *be a Kawamata log-terminal pair of relative dimension* 2 *over a discrete valuation ring* R *with residue field* k *and fraction field* K *such that* k *is algebraically* *closed of characteristic $p > 0$. Assume $K_X + \Delta$ is pseudo-effective and simple normal crossings over R. Then the numerical Kodaira dimensions satisfy*

$$\nu\left(K_{X_K} + \Delta_K\right) = \nu\left(K_{X_k} + \Delta_k\right).$$

As a result of abundance for log surfaces, the log Kodaira dimensions also satisfy

$$\kappa\left(K_{X_K} + \Delta_K\right) = \kappa\left(K_{X_k} + \Delta_k\right).$$

3.2 Background

Following the notation of [Suh08], let p be a prime number and let R be a discrete valuation ring with residue field k and fraction field K, such that K is either of characteristic 0 or p. For a scheme Z over R, Z_k will denote the special fiber $Z \otimes_R k$ and Z_K will denote the generic fiber $Z \otimes_R K$.

Katsura and Ueno's lemma on the deformation of a negative extremal curve is a main ingredient in the proof of theorem 3.3. I include this proof below almost verbatim except for noting that the residue field is unchanged after the extension in the proof. First a technical lemma which allows me to claim this:

Lemma 3.5. *Let $f : X \to S$ be a locally finite type morphism with $S = Spec\ R$ where R is a discrete valuation ring. Let $s \in S$ and $x \in X_s = X \times_S Spec\ k(s)$. Suppose f is smooth at x. Then there exists a discrete valuation ring $\tilde{R} \supset R$ and $s\prime \in Spec\tilde{R}$ with finite residue field extension $k(s')/k(x)$ and a morphism $j : Spec(\tilde{R}) \to X \times_S S' = Y$ over $Spec(R)$ with $j(s') = x$.*

Proof. (Following [BLR12, 2.2.14]). Let n be the relative dimension of X over s at x. Let $\mathcal{J}_x \subset \mathcal{O}_{X_s}$ be the sheaf of ideals associated to the closed point x of X_s. As f is smooth at x, $Spec\ k(x) \to Spec\ k(s)$ is étale. Thus $\mathcal{J}_x$ is generated by n elements $\overline{g}_1, ..., \overline{g}_n$ such that their differentials $d\overline{g}_1, ..., d\overline{g}_n$ generate $\Omega'_{X/S} \otimes k(x)$. By the Jacobi Criterion [BLR12, 2.2.7], smoothness at x is an open condition. Thus $\overline{g}_1, ..., \overline{g}_n$ lift to sections $g_1, ..., g_n$ of $\mathcal{O}_X$ defined on an open neighborhood of $x \in X$. Let S' be the subscheme of X defined by $g_1, ..., g_n$. By the Jacobi Criterion, S' is étale over S at x. After shrinking s', we may assume (since étale-ness is an open condition) that $S' \to S$ is étale. The tautological section $h' : S' \to X'$ is then a section as required. ☐

Lemma 3.6. *[KU85, 9.4] Let $\varphi : X \to Spec\,(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $Spec\,(R)$, where R is a discrete valuation ring with* *algebraically closed residue field k, and field of fractions K. Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ. If X_k contains an exceptional curve of the first kind e, there exists a discrete valuation ring $\tilde{R} \supset R$, with residue field isomorphic to k, and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \to \tilde{Y}$ over $Spec\left(\tilde{R}\right)$ such that on the closed fibre, π induces the contraction of the exceptional curve e. Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.*

Proof. By **[Art69, Cor 6.2]** $Hilb_{X/Spec(R)}$ is represented by an algebraic space $\mathscr{H}$ which is locally of finite type over $Spec\,(R)$. Let Y be the irreducible component containing the point $\{e\}$ corresponding to the exceptional curve e on the special fiber. Then $e \approx \mathbb{P}^1_k$ and $N_{e/X} \approx \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, so Y is regular at $\{e\}$ and of dimension 1. Since e is fixed in the special fiber, the structure morphism $p : Y \to Spec\,(R)$ is surjective. By Lemma 3.5, we can find an étale cover $\tilde{R} \supset R$ and a morphism $j : Spec\left(\tilde{R}\right) \to Y$ over $Spec\,(R)$ with $j\,(\tilde{o}) = \{e\}$ (if R is not already complete, then first extend R to a complete discrete valuation ring using [GD71, 0.6.8.2,3] so that $\tilde{R}$ is again a discrete valuation ring). As k is assumed algebraically closed, and $\tilde{R} \to R$ is unramified, then the extension of residue fields is finite and separable at the closed point of R, and hence an isomorphism of residue fields. The rest of the proof proceeds as in the source. □

By the above lemma, it is possible, in certain cases, to reduce the minimal model program for an arithmetic threefold over a discrete valuation ring to the minimal model program on the special fiber. The following technical ingredients of the minimal model program for surfaces in positive characteristic will therefore be useful:

Lemma 3.7. *[KK94, 2.3.5] Let (S, B) be a log-canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C.(K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

Theorem 3.8. *[Tan14, 5.3] Let X be a projective normal surface and let C be a curve in X such that $r\,(K_X + C)$ is Cartier for some positive integer r.*

1. *If $C \cdot (K_X + C) < 0$, then $C \approx \mathbb{P}^1$.*
2. *If $C \cdot (K_X + C) = 0$, then $C \approx \mathbb{P}^1$ or $\mathcal{O}_C\left((K_X + C)^{[r]}\right) \approx \mathcal{O}_C$.*

In addition to the vanishing theorem 2.14 mentioned in Chapter 2, the following vanishing theorem may be used for cohomological arguments when there is no boundary divisor.

Theorem 3.9. *[Eke88, II.6] Let X be a minimal surface of general type and let $\mathcal{L}$ be an invertible sheaf that is numerically equivalent to $\omega_S^{\otimes i}$ for some $i \geq 1$. Then $H^1\left(X, \mathcal{L}^{\otimes j}\right) = 0, j \geq 2$ except possibly for certain surfaces in characteristic 2 with $\chi\left(\mathcal{O}_X\right) \leq 1$.*

Since not all arithmetic threefolds are over domains with algebraically closed residue fields where the above lemma was proven, (in fact $\mathbb{Z}$, arguably the most important case has finite residue fields) it will be necessary to perform a base-change. The following properties of base change will be used:

Theorem 3.10. *[Har77, III.10.2] Let $f : X \to Y$ be a morphism of schemes of finite type over a field k. Then f is smooth of relative dimension n if and only if:*

1. *f is flat; and*
2. *for each point $y \in Y$, let $X_{\overline{y}} = X_y \otimes_{k(y)} k\left(y\right)^-$ where $k\left(y\right)^-$ is the algebraic closure of $k\left(y\right)$.*

Then $X_{\overline{y}}$ is equidimensional of dimension n and regular. (We say the "fibres of f are geometrically regular of equidimension n.")

Theorem 3.11. *[GW10, 6.28] Let k be a field, X a k-scheme locally of finite type, and let $x \in X$ be a closed point. Let $d \geq 0$ be an integer. We fix an algebraically closed extension K of k, and write $X_K = X \otimes_k K$. The following are equivalent:*

1. *The k-scheme X is smooth of relative dimension d at x.*
2. *For every point $\overline{x} \in X_K$ lying over x, X_K is smooth of relative dimension d at $\overline{x}$.*
3. *For every point $\overline{x} \in X_K$ lying over x, the completed local ring $\hat{\mathcal{O}}_{X_K,\overline{x}}$ is isomorphic to a ring of formal power series $K[[T_1, ..., T_d]]$ over K.*
4. *For every point $\overline{x} \in X_K$ lying over x, the local ring $\mathcal{O}_{X_K,\overline{x}}$ is regular and has dimension d.*
5. *The equalities $dim_{\kappa(x)} T_x\left(X/k\right) = dim \mathcal{O}_{X,x} = d$ hold.*

If these conditions are satisfied, then the local ring $\mathcal{O}_{X,x}$ is regular and has dimension d. Furthermore, if $\kappa\left(x\right) = k$, then the final condition implies the other ones.

As a consequence of the above theorems, I note that as long as the residue field is perfect, the singularities of the pair are preserved by base change:

Proposition 3.12. *Let (X, Δ) be a terminal (resp. Kawamata log-terminal) pair which is simple normal crossings over a discrete valuation ring R with perfect residue field k and perfect fraction field K. Let (X', Δ') denote the pair obtained by base change to a complete discrete valuation ring R' with algebraically closed residue field. Then (X'_k, Δ'_k) and (X'_K, Δ'_K) are log-smooth and terminal (resp. Kawamata log-terminal.)*

Proof. By adjunction and log smoothness (X_k, Δ_k) and (X_K, Δ_K) are terminal (resp. Kawamata log-terminal) and simple normal crossings by definition 2.3. Since smoothness is preserved by base-change, then all strata of (X', Δ') are smooth. Also, (X'_k, Δ'_k) is by definition log smooth after base change to the algebraic closure of k, since the algebraic closure of a perfect field is equal to its separable closure. Thus (X', Δ') is log-smooth. Next, since k is assumed perfect, the notions of smooth and regular coincide. Thus by Theorems 3.11, and 3.10 we have that the base change $X \to X'$ over the algebraic closure k' is smooth of relative dimension 0, hence an étale base change. Thus by [Kol13, 2.14,2.15], (X_k, Δ_k) and (X_K, Δ_K) are terminal (resp. Kawamata log-terminal) iff (X'_k, Δ'_k) and (X'_K, Δ'_K) are terminal (resp. Kawamata log-terminal). □

Finally, I note two well-known results which describe how cohomology behaves in a family.

Theorem 3.13. *[Liu02, 5.3.20, 5.3.22] Let $S = Spec\ \mathcal{O}_K$ be the spectrum of a discrete valuation ring $\mathcal{O}_K$, with generic point η and closed point s. Let $f : X \to S$ be a projective morphism and $\mathcal{F}$ a coherent sheaf on X that is flat over S. Fix $p \geq 0$. TFAE.*

1. *We have equality $dim_{k(s)} H^p(X_s, \mathcal{F}_s) = dim_{k(\eta)} H^p(X_\eta, \mathcal{F}_\eta)$.*
2. *$H^p(X, \mathcal{F})$ is free over $\mathcal{O}_K$ and the canonical homomorphism $H^p(X, \mathcal{F}) \otimes_{\mathcal{O}_K} k(s) \to H^p(X_s, \mathcal{F}_s)$ is a bijection.*
3. *The $\mathcal{O}_K-$module $H^{p+1}(X, \mathcal{F})$ is torsion-free.*

Also, regardless of whether the above hold, $\chi_{k(s)}(\mathcal{F}_s) = \chi_{k(\eta)}(\mathcal{F}_\eta)$.

Theorem 3.14. *[Har77, III.12.11] Let $f : X \to Y$ be a projective morphism of Noetherian Schemes, and let F be a coherent sheaf on X, flat over Y. Let y be a point of Y. Then*

1. *if the natural map $\varphi^i(y) : R^i f_*(F) \otimes k(y) \to H^i(X_y, F_y)$ is surjective, then it is an isomorphism, and the same is true for all y' in a suitable neighborhood of y.*
2. *Assume $\varphi^i(y)$ is surjective. Then the following are equivalent:*

(i) $\varphi^{i-1}(y)$ *is also surjective.*

(ii) $R^i f_*(F)$ *is locally free in a neighborhood of* y*.*

3.3 Proof of the Invariance of Plurigenera

The proof of this chapter's main result goes in several steps. The first step is to show under more restrictive assumptions on the base locus of the pair (X, Δ), that the invariance of plurigenera holds. The base locus of a divisor roughly describes the part of that divisor which won't move in a linear system. The "diminished base locus," is similar, but an ample divisor is added. This is helpful for studying the minimal model program, since it gives the non-nef part of the divisor.

Definition 3.15. [Laz04] Let D be a pseudoeffective $\mathbb{R}$-divisor on a normal projective variety X. The diminished (stable) base locus is defined as

$$\mathbb{B}_-(D) := \bigcup_{\substack{A \;\, Ample \\ D+A \;\, \mathbb{Q}-Cartter}} \mathbb{B}(D+A)$$

where $\mathbb{B}(D+A) = \bigcap_{n\geq 1} Bs(n(D+A))$ is the stable base locus.

One way to simplify the base locus of $K_X + \Delta$ is to remove part of it's diminished base locus. In terms of divisors, this can be done by subtracting N_σ. Note that on a surface, an effective divisor can be separated into its "nef" part and its "non-nef" part (this decomposition is called the Zariski Decomposition, and doesn't always exist in higher dimensions), and in the surface case, N_σ corresponds exactly with the "non-nef" part.

Definition 3.16. Let D be a big Cartier divisor. Let F_m be the fixed divisor $|mD|_{fix}$. Then $F_{m+n} \leq F_m + F_n$ and the limit

$$N_\sigma(D) := \lim_{m\to\infty} \frac{1}{m} F_m$$

exists as an $\mathbb{R}$-divisor. We have $mN_\sigma(D) \leq F_m$ and for the $\mathbb{R}$-divisor $P_\sigma(D) := D - N_\sigma(D)$, we have an isomorphism

$$H^0(X, \mathcal{O}_X(mD)) \approx H^0(X, \mathcal{O}_X(\llcorner mP_\sigma(D)\lrcorner))$$

for any $m > 0$. The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the sectional decomposition.

Remark 3.17. If D is not big, then $|mD|$ may be empty for certain positive integers m, and thus, in defining $N_\sigma(D)$, it is necessary to consider only the semigroup $\mathbb{N}(D)$ of m such that $|mD| \neq \emptyset$ for such D. In this chapter, only the sectional decomposition of big divisors is considered.

First I prove a special case of Theorem A.

Proposition 3.18. *Let (X, Δ) be a terminal log smooth pair of relative dimension 2 over a discrete valuation ring R with algebraically closed residue field k of characteristic $p > 2$ and perfect fraction field K. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\nu(K_X + \Delta) \neq 1$. Assume that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$. Then there exists an m_0 such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

The proof of proposition 3.18, is given in the following claims.

Claim 3.19. Assumptions as above, after passing to an extension R' of R, there is a proper, smooth algebraic space X^{min}/R' and an R' morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef.

Proof. As k is algebraically closed of characteristic $p > 0$, then by the Cone Theorem 2.12,

$$\overline{NE}(X_k) = \overline{NE}(X_k)_{K_{X_k}+\Delta_k \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

with each C_i is rational or $C_i = B_j$ for some B_j a component of Δ with $B_j^2 < 0$. Under the assumption that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$, then actually each C_i is rational and is not a component of Δ_k. Thus $C_i.\Delta_k \geq 0$ so $C_i.K_{X_k} < 0$, and since $N_\sigma(K_{X_k} + \Delta_k)$ is a $\mathbb{Q}$-divisor (applying the finite generation of the log-canonical ring on the special fiber c.f. [Tan14, 7.1]), then Theorem 3.8 implies $C_i \approx \mathbb{P}^1$. By Lemma 3.7, C_i can be contracted to a log-canonical point, which is actually a smooth point under the terminal assumption. Thus C_i is an exceptional curve of the first kind, so it is possible to apply Lemma 3.6.

By Lemma 3.6, there is a discrete valuation ring $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \to \tilde{X}$ induces the contraction of C_i on X_k and X_K, and further $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since k is algebraically closed, it induces identity on residue fields. Now I need to work with $\tilde{X}$. In order to apply the Cone Theorem 2.12 again, I need $\tilde{X}_k$ to be projective, but a smooth algebraic space of dimension 2, proper, separated, and of finite type over an algebraically closed field is projective, so $\tilde{X}_k$ is projective. Thus the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no problem in extending R. As the Picard number of X_k drops at each step, there are only finitely many steps. □

Claim 3.20. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef.

Proof. It suffices to apply 2.11 to X_k^{min}, and then restrict to X_K. □

Now we proceed by cases depending on the Kodaira dimension.

Claim 3.21. (Case 1: $\nu = 2$) In the case that $\nu(K_{X_k} + \Delta_k) = 2$, there is an m_0 such that for $m_0|m$, we have

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Proof. By the above, we achieve X^{min} such that $K_X + \Delta$ is nef on both X_k^{min} and X_K^{min}. By the Theorem 2.14 applied to the special fiber (and the semicontinuity theorem) there exists an $m_0 \gg 0$ such that for $m_0 < m$ and $i > 0$, we have

$$0 \leq h^i\left(m\left(K_{X_K} + \Delta_K\right)\right) \leq h^i\left(m\left(K_{X_k} + \Delta_k\right)\right) = 0.$$

In fact, if $\Delta = 0$ and $p > 2$, then Ekedahl's vanishing Theorem 3.9 can be applied. Thus, applying the invariance of Euler characteristic (Theorem 3.13), and birational invariance of the plurigenera, it follows that for $m > m_0$,

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

□

Claim 3.22. (Case 2: $\nu = 0$) In the case that the special fiber has $\nu\left(K_{X_k} + \Delta_k\right) = 0$, there is an m_0 such that for $m_0|m$, we have

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Proof. As before, we achieve an $\pi' : X^{min} \to R'$ and an R' morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and both $K_{X_k^{min}} + \Delta_{X_k^{min}}$ and $K_{X_K^{min}} + \Delta_{X_K^{min}}$ are nef. Applying [Tan14], we find that $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is semi-ample. Under the pseudo-effective assumption, and since $\nu\left(K_{X_k^{min}} + \Delta_{X_k^{min}}\right) = 0$, but $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef, then actually $K_{X_k^{min}} + \Delta_{X_k^{min}} \equiv 0$. But then $K_{X_k} + \Delta_k \sim_{\mathbb{Q}} 0$. The same holds on the generic fiber, since the abundance has been proven over a field k in [Tan15a]. Thus, taking $m' = m_k \cdot m_K$, the conclusion follows. □

3.4 Invariance: General Case

The next step is to remove the restriction on the base locus and residue field from proposition 3.18.

Theorem 3.23. *Let (X, Δ) be a Kawamata log-terminal pair of relative dimension 2 over a discrete valuation ring R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is, big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists an m_0 depending on the intersection numbers, such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Proof. Since the hypothesis and conclusion are preserved by base change, then extending R, we may assume that k is algebraically closed and that R is complete. The proof follows similarly to [HMX13, 1.6] except that Proposition 3.18 is used in place of Kawamata Viehweg Vanishing. Replacing (X, Δ) by a blow-up, assume (X, Δ) is terminal. Recall that the log-canonical ring of $K_{X_k} + \Delta_k$ is finitely generated (c.f. [Tan14, 7.1]) so $N_\sigma\left(K_{X_k} + \Delta_k\right)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma\left(K_{X_k} + \Delta_k\right)$$

and let $0 \leq \Theta \leq \Delta$ be a $\mathbb{Q}$-divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. By definition 3.16, letting $m \gg 0$ to fit the hypothesis of Proposition 3.18, and sufficiently divisible such that $m\left(K_{X_k} + \Theta_k\right)$ is integral, then

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_k} + \Theta_k\right)\right)$$

and furthermore, by Proposition 3.18,

$$h^0\left(m\left(K_{X_k} + \Theta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Theta_K\right)\right).$$

As $\Theta \leq \Delta$, the theorem follows by semicontinuity. □

A simple, but useful, consequence is the invariance of Kodaira dimensions.

Corollary 3.24. *Let (X, Δ) be a Kawamata log-terminal log-smooth pair of relative dimension 2 over a discrete valuation ring R with residue field k and fraction field K such that*

k is perfect. Assume $K_X + \Delta$ is pseudo-effective. Then the numerical kodaira dimension satisfy

$$\nu\left(K_{X_K} + \Delta_K\right) = \nu\left(K_{X_k} + \Delta_k\right).$$

As a result of abundance for log surfaces, the log kodaira dimension also satisfy

$$\kappa\left(K_{X_K} + \Delta_K\right) = \kappa\left(K_{X_k} + \Delta_k\right).$$

Proof. For any ample A, there exists an m_0 such that for $m > m_0$, $K_X + \Delta + \frac{1}{m}A = K_X + \Delta'$ is Kawamata log-terminal and big (since $K_X + \Delta$ is assumed pseudo-effective). Thus the result follows from Theorem 3.23. $\square$

CHAPTER 4

GENERATION AND ABUNDANCE

In this chapter, I use the Invariance of Plurigenera (Theorem 3.23) to show finite generation of two different types of log-canonical rings and, as a consequence, derive a form of the abundance theorem. In characteristic 0, finite generation of the log-canonical ring for Kawamata log-terminal pairs is known by [BCHM10] in all dimensions. Whether adjoint rings are finitely generated is still an open question, and the proof of even a seemingly fairly weak version of this would imply the existence of minimal models in general (c.f. [CL13]) (A fact I use advantageously in Chapter 5). In positive characteristic, and dimension ≤ 3, the finite generation seems to be known in characteristics $p > 5$ by [Bir15]. In this chapter, I prove the mixed characteristic, general type case which also holds for a family of surfaces in equal characteristic.

An "adjoint" ring is a generalization of a log-canonical ring defined as follows:

Definition 4.1. [CZ14, 2.8] Let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. D_i are called **adjoint divisors** on X if they are of the form $D_i = K_X + \Delta_i$ for some pair (X, Δ_i) where X is normal and projective, $\Delta_i \geq 0$ and is a $\mathbb{Q}$-divisor. Let

$$\mathfrak{R} = R\left(X; D_1, ..., D_r\right) = \bigoplus_{(n_1,...,n_r)\in\mathbb{N}^r} H^0\left(X, n_1D_1 + \cdots + n_rD_r\right).$$

The ring $\mathfrak{R}$ is said to be "generated in degree m" if it is generated by sections of

$$H^0(X, \mathcal{O}_X(\sum_{i=1}^{n} a_iD_i))$$

with $a_1, ..., a_k \in \{0, ..., m\}$. A typical way to show this is to show that, for any $m_1, ..., m_k \geq 0$ and at least one $m_\ell > m$ and $G = \sum_{i=0}^{k} m_iD_i$, then the following multiplication map is surjective:

$$H^0(G - D_\ell) \otimes H^0(D_\ell) \to H^0(G).$$

Note this implies that the individual rings $R(D_i)$ are finite generated as well.

Typical D_i are divisors of the form $K_X + \Delta$ where (X, Δ) is a pair. If there is just one D_i, $D_1 = K_X + \Delta$, the above is called the "log-canonical ring," or just the canonical ring if $\Delta = 0$.

The finite generation of these objects is important to the minimal model program as it is used to prove the existence of birational contractions of extremal rays, which allows you to take a step in the Minimal Model Program closer to the minimal model as discussed in Chapter 2.

4.1 Finite Generation (Big Case)

First I recall two versions of the famous lemma due to Nakayama:

Lemma 4.2. *[GH94, Chapter 5.3] Let M, N be modules over a discrete valuation ring R with residue field $R/\mathfrak{m}R$. Then*

1. *A minimal set of generators of M restricts to a basis for $M/\mathfrak{m}M$ and conversely a basis for $M/\mathfrak{m}M$ extends to a minimal set of generators of M.*
2. *If $f : M \to N$ induces a surjective morphism $\overline{f} : M/\mathfrak{m}M \to N/\mathfrak{m}N$, then f is surjective.*

Theorem 4.3. *Let (X, Δ) be a big, Kawamata log-terminal log-smooth pair of relative dimension 2 over a discrete valuation ring R with residue field k and fraction field K such that k is perfect. Then the log-canonical ring $R(K_X + \Delta)$ is finitely generated over R.*

Proof. By [Tan14], there is $m \gg 0$, such that

$$\bigoplus_{m \geq 0} H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right)$$

is generated in degree 1, so that there is a surjection

$$S^l H^0\left(m\left(K_{X_k} + \Delta_k\right)\right) \twoheadrightarrow H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right).$$

Consider the following diagram:

$$\begin{tikzcd}
S^l H^0\left(m\left(K_X + \Delta\right)/R\right) \arrow[r, "\alpha"] \arrow[d] & H^0\left(ml\left(K_X + \Delta\right)/R\right) \arrow[d] \\
S^l H^0\left(m\left(K_{X_k} + \Delta_k\right)\right) \arrow[r, "\twoheadrightarrow"] & H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right)
\end{tikzcd}$$

By Theorems 3.23 and 3.14,

$$H^0(n(K_{X_k} + \Delta_k)) = H^0(n(K_X + \Delta))/\mathfrak{m}R$$

for all sufficiently divisible n. Thus, by Lemma 4.2(1), the vertical maps are surjective. Now applying part (2) of Lemma 4.2, surjectivity of the bottom map implies surjectivity of α, and thus $R(K_X + \Delta)$ is finitely generated over R. $\square$

The above theorem uses finite generation at the special fiber to deduce finite generation over the discrete valuation ring by applying Theorem 3.23. In the rest of the Chapter, I will use this technique to do the same for adjoint rings, and in addition study the problem over a Dedekind Domain.

4.2 Finiteness of Models For Surfaces

The following theorem is not new (in fact I pull the proof essentially verbatim from the characteristic 0 case), but is stated here as the minimal model program is not usually run with scaling on a surface (in [KK94, Tan14], termination is proven without scaling). The theorem will be used once I have reduced the general case of termination to the minimal model program on the special fiber.

Theorem 4.4. *Let X be a two dimensional normal variety over a perfect field of characteristic $p > 0$. Let (X, Δ) be a log smooth KLT pair with a good minimal model (which exists for non-negative Kodaira dimensions by [Tan14]) and let A be an ample $\mathbb{Q}$-divisor on X. Then there exists an $\epsilon > 0$ such that the minimal models (and the output of the minimal model program with scaling) for $(X, \Delta + tA)$ are all isomorphic for $t \in [0, \epsilon]$.*

Proof. (Essentially same as [Lai11, 26,27] and simplifies due to the fact that a birational isomorphism in codimension 1 on a surface gives an isomorphism). Let $\phi : (X, \Delta) \to (X_g, \Delta_g)$ be a good minimal model. Let $f : X_g \to Z = Proj\ R\left(K_{X_g} + \Delta_g\right)$ the contraction (which exists by since $K_{X_g} + \Delta_g$ is nef, hence semiample by [Tan14]). Then ϕ contracts the divisorial part of $\mathbb{B}(K_X + \Delta)$. Pick $t_0 > 0$ such that

$$(X_g, \Delta_g + t_0\phi_* A) = (X_g, \Delta_g + t_0 A_g)$$

is Kawamata log-terminal with A ample on X (note A_g is big, not in general nef). For H ample on X_g, let $\phi : X_g \to X'$ be the minimal model program with scaling, which terminates (again by [Tan14]), and gives a minimal model of $(X_g, \Delta_g + t_0 A_g)$ over Z. For any curve contracted by f, $\left(K_{X_g} + \Delta_g\right).C = 0$, hence $K_{X'} + \psi_*\Delta_g = K_{X'} + \Delta' \equiv_Z 0$. Thus curves contracted by ψ have trivial intersection with $K_{X_g} + \Delta_g$, and intersect negatively with A_g. Thus changing t does not affect which curves intersect negatively with $K_{X_g} + \Delta_g + tA_g$, and so X' is a minimal model, over Z, of $(X_g, \Delta_g + tA_g)$ for all $t \in (0, t_0]$.

Now $\Delta' + t_0A'$ with $A' = \psi_* A_g$ (which is big) implies by the Theorem 2.12, that there exist only finitely many $K_{X'} + \Delta' + t_0A'$ negative extremal rays in $\overline{NE}\left(X'\right)$. These are all necessarily just intersecting A' negatively, so decreasing t_0, eventually we get to a point where a further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (now we are on X', not over Z). Again shrinking t_0, suppose that $\psi \circ \phi$ is discrepancy negative with respect to $(X, \Delta + tA)$ for all t in the half open interval $(0, t_0]$. Note that $\mathbb{B}\left(K_X + \Delta + t_0A\right) \subset \mathbb{B}\left(K_X + \Delta\right)$, so ψ contracts only the curves not contracted by ϕ. Thus $\psi \circ \phi$ is discrepancy negative on the closed interval $[0, t_0]$, so X' is a minimal model of $(X, \Delta + tA)$ for all $t \in [0, t_0]$. Thus $\mathbb{B}\left(K_X + \Delta + tA\right)$ has the same divisorial components for all $t \in [0, t_0]$. Hence, any two minimal models for different $t \in [0, t_0]$ are birational and isomorphic in codimension one, and as the total dimension is two, they are thus isomorphic. ☐

Applying the Theorem 4.4 to the proof of Theorem 3.23, I can achieve finite generation of a more generalized type of log-canonical ring:

Theorem 4.5. *Let (X, Δ) be a big Kawamata log-terminal log-smooth pair of relative dimension 2 over a discrete valuation ring R with residue field k and fraction field K such that k is perfect. Let A be a basepoint-free ample divisor. Then there exists $t > 0$ such that the adjoint ring*

$$R(X, K_X + \Delta, K_X + \Delta + tA)$$

is finitely generated over R.

Proof. Since the hypothesis and conclusion are preserved by base change, then extending R, we may assume that k is algebraically closed and that R is complete. Replacing (X, Δ) by a blow-up, assume (X, Δ) is terminal. Recall that the log-canonical ring of $K_{X_k} + \Delta_k$ is finitely generated (c.f. [Tan14, 7.1]) so $N_\sigma\left(K_{X_k} + \Delta_k\right)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma\left(K_{X_k} + \Delta_k\right)$$

and let $0 \leq \Theta \leq \Delta$ be a $\mathbb{Q}$-divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. As in the proof of Proposition 3.18, we achieve X^{min} such that $K_X + \Delta$ is nef on both X_k^{min} and X_K^{min}. By Theorem 4.4 and 2.11, $K_X + \Delta + tA$ is also nef on both fibers when t is sufficiently small. Thus the higher cohomologies of any $\mathbb{Z}$-linear, integral combination D

of $K_X + \Delta$ and $K_X + \Delta + tA$ vanish. As in the proof of Theorem 3.23, this implies the invariance:

$$H^0(X_k, D_k) = H^0(X_K, D_K)$$

By abundance on X_k (which holds by [Tan14, Tan15a]), some high multiple of such D_k is base-point free. (In the next two sections, I will study which multiple is base-point free).

By [HK00b], the ring

$$R(K_{X_k} + \Delta_k, K_{X_k} + \Delta_k + tA_k)$$

is finitely generated. (In fact, one can be more precise about the degrees of generation). Now applying definition 4.1, the remainder of the proof follows easily by pulling back, using Lemma 4.2, the necessary surjective multiplication maps as in Theorem 4.21.

□

4.3 Big Abundance / R

The "Abundance" theorem is a key step in the minimal model program. It can be reduced down to showing that the Kodaira Dimensions are equal [Nak04], or the statement that nefness of an adjoint divisor implies some high multiple is basepoint free. This is useful since basepoint-freeness of a collection of adjoint divisors can be used to easily imply finite generation of their adjoint ring (c.f. the proof of Theorem 4.21). First some technical definitions related to adjoint rings.

Definition 4.6. Suppose X is an arithmetic threefold. Let Γ a geometric valuation on X. Define

$$\sigma_\Gamma = inf \left\{ mult_\Gamma D' \mid D \sim_{\mathbb{R}} D' \geq 0 \right\} \in \mathbb{R}.$$

Definition 4.7. Let k be either $\mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$. A k-divisor D is k-**effective** if there is an effective divisor $D' \geq 0$ which is k-linearly equivalent to D. The set of such divisors is denoted $Div_k^{eff} \subset \overline{Eff}(X)$. Let $D_1, ..., D_r$ be adjoint divisors on X with adjoint ring $\mathfrak{R} = R(X, D_1, ..., D_r)$. The **support** of $\mathfrak{R}$ is defined to be

$$Supp\ \mathfrak{R} = \left(\sum_{i=1}^{r} \mathbb{R}_+ D_i \right) \cap Div_{\mathbb{R}}^{eff}(X).$$

The following result is the basis for reducing both the Cone Theorem and Termination of Flips in the minimal model program to finite generation of an adjoint ring.

Theorem 4.8. *Let X be a normal scheme of relative dimension 2, proper over R. Let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier divisors on X. Assume that $R(X; D_1, ..., D_r)$ is finitely generated. Let $D : \mathbb{R}^r \ni (\lambda_1, ..., \lambda_r) \mapsto \sum \lambda_i D_i \in Div_{\mathbb{R}}(X)$ be the tautological map.*

(1) The support of R is a rational polyhedral cone.

(2) If Supp R contains a big divisor and $D \in \sum \mathbb{R}_+ D_i$ is pseudoeffective, then $D \in$ Supp R.

(3) There is a finite rational polyhedral subdivision Supp $R = \bigcup C_i$ such that σ_Γ is a linear function on C_i for every geometric valuation Γ of X. Furthermore, there is a coarsest subdivision with this property, in the sense that, if i and j are distinct, there is at least one geometric valuation Γ of X such that (the linear extensions of) $\sigma_\Gamma|_{C_i}$ and $\sigma_\Gamma|_{C_j}$ are different.

(4) There is a finite index subgroup $\mathbb{L} \subset \mathbb{Z}^r$ such that for all $n \in \mathbb{N}^r \cap \mathbb{L}$, if $D(n) \in$ Supp R, then $\sigma_\Gamma(D(n)) = mult_\Gamma |D(n)|$.

Proof. [CL13, 3.5] Parts (1) and (2) follow easily from the cited theorem. For (3) and (4) note that [ELM⁺06, 4.7] is stated for an arbitrary Noetherian scheme, and this theorem is the only result used by the proof given in [CL13, 3.5]. ☐

Theorem 4.9. *([CL13, 3.6]) Let (X, Δ) be a Kawamata log-terminal pair of relative dimension 2 over R. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and that $R(K_X + \Delta, K_X + \Delta + A)$ is finitely generated. If $K_X + \Delta$ is pseudoeffective, then it is $\mathbb{Q}$-effective.*

Proof. The hypothesis include the requirements to apply Theorem 4.8. By part (2) of that Theorem, $K_X + \Delta$ is in

$$Supp\ R(K_X + \Delta, K_X + \Delta + A)$$

which recall, by definition 4.7 means that $K_X + \Delta$ is in $Div^{eff}_{\mathbb{R}}(X)$, and since it is already assumed $\mathbb{Q}$-Cartier, this completes the proof. ☐

As a result of Theorem 4.21, there is the following abundance theorem:

Corollary 4.10. *Let (X, Δ) be a big, Kawamata log-terminal pair of relative dimension 2 over R. If $K_X + \Delta$ is nef, then it is semi-ample.*

Proof. (After applying Theorem 4.5, this follows similarly to [CL13, 3.8]). Let A be a basepoint-free ample divisor on X with $K_X + \Delta + A$ Kawamata log-terminal. For any $\epsilon > 0$, and any geometric valuation Γ on X, as $K_X + \Delta + \epsilon A$ is ample, a positive multiple is basepoint free. Thus,

$$\sigma_\Gamma\left(K_X+\Delta+\epsilon A\right)=0$$

so that, after applying Theorem 4.5, σ_Γ is identically zero by Theorem 4.8(4). Therefore, the centre of Γ is not in $\mathbb{B}\left(K_X+\Delta\right)$ for any such Γ. □

Corollary 4.11. *Let X be a normal projective scheme of relative dimension 2 over R and let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. Assume that $\mathfrak{R}=R(X,D_1,...,D_r)$ is finitely generated, and let Supp $\mathfrak{R}=\bigcup\mathcal{C}_i$ be a finite rational polyhedral subdivision such that for every geometric valuation Γ of X, σ_Γ is linear on $\mathcal{C}_i$, as in Theorem 4.8. Denote $\varphi:\sum\mathbb{R}_+D_i\to N^1\left(X\right)_\mathbb{R}$ the projection, and assume there exists k such that $\mathcal{C}_k\cap\varphi^{-1}\left(Amp\ D\right)\neq\emptyset$. Then $\mathcal{C}_k\subset$ Supp $\mathfrak{R}\cap\varphi^{-1}\left(Nef\ X\right)$. If the subdivision is coarsest, then $\mathcal{C}_k=$ Supp $\mathfrak{R}\cap\varphi^{-1}\left(Nef\ X\right)$*

Proof. The asymptotic order function σ_Γ is zero on any ample divisor (since by definition, a high multiple has no base locus). By assumption $\mathcal{C}_k\cap\varphi^{-1}(Amp\ X)\neq\emptyset$. Thus the asymptotic order functions σ_Γ are zero on a nonempty subset of $\mathcal{C}_k$. As $\mathfrak{R}$ is finitely generated, Theorem 4.8(3) applies so that σ_Γ is linear. For any divisor D in $\mathcal{C}_k$, take B sufficiently ample so that $D+B$ is ample. Then

$$0=\sigma_\Gamma(D+B)=\sigma_\Gamma(D)+\sigma_\Gamma(B)=\sigma_\Gamma(D)+0=\sigma_\Gamma(D)$$

Any $D\in\mathcal{C}_k$ is $\mathbb{R}$-linearly equivalent to a $\mathbb{Z}$-combination of the D_i so that $R(D)$ is finitely generated by hypothesis (see definition 4.1). This implies, by Theorem 4.8(4) (c.f. [CL13, 3.7(2)]), that the centre of Γ is not in $\mathbb{B}(D)$ for any such Γ, and thus $\mathbb{B}(D)=\emptyset$, so that D is semiample, hence nef. The final statement follows exactly as in the source. □

4.4 KLT Vanishing

In this section I study the computability of the constant given in Theorem 2.14. The reason for this is to show that the constant achieved in Theorem 3.23 is actually characteristic free (possibly after throwing away finitely many characteristics). This will then be applied to the proof of a generalized version of Theorem 4.3.

First I recall an effective vanishing theorem which applies to surfaces in positive characteristic:

Theorem 4.12. *[Ter99] Let X be a smooth projective surface over an algebraically closed field of characteristic $p>0$ and let D be a big and nef Cartier divisor on X. Assume that either*

1. *$\kappa\left(X\right)\neq2$ and X is not quasi-elliptic with $\kappa\left(X\right)=1$; or*

2. *X is of general type with $p \geq 3$ and $\left(D^2\right) > vol\,(X)$ or $p = 2$ and*

$$\left(D^2\right) > max\left\{vol\,(X)\,, vol\,(X) - 3\chi\left(\mathcal{O}_X\right) + 2\right\}.$$

Then

$$H^i\left(X, \mathcal{O}_X\left(K_X + D\right)\right) = 0$$

for all $i > 0$.

Recall the following covering Lemma which Tanaka uses in the proof of Theorem 2.14.

Lemma 4.13. *[KMM87, Theorem 1.1] Let X be an n-dimensional smooth variety. Let D be a $\mathbb{Q}$-divisor such that the support of the fractional part $\{D\}$ is simple normal crossing. Moreover, suppose that, for the prime decomposition $\{D\} = \sum_{i\in I} \frac{b^{(i)}}{a^{(i)}} D^{(i)}$, no integers $a^{(i)}$ are divisible by p. Then there exists a finite surjective morphism $\gamma : Y \to X$ from a smooth variety Y with the following properties:*

- *The field extension $K(Y)/K(X)$ is a Galois extension.*
- *$\gamma^* D$ is a $\mathbb{Z}$-divisor.*
- *$\mathcal{O}_X\left(K_X + \ulcorner D \urcorner\right) \approx \left(\gamma_* \mathcal{O}_Y\left(K_Y + \gamma^* D\right)\right)^G$, where G is the Galois group of $K(Y)/K(X)$.*
- *If D' is a $\mathbb{Q}$-divisor such that $\{D'\} = \{D\}$,*

then $\gamma^* D'$ is a $\mathbb{Z}$-divisor, and $\mathcal{O}_X\left(K_X + \ulcorner D' \urcorner\right) \approx \left(\gamma_* \mathcal{O}_Y\left(K_Y + \gamma^* D'\right)\right)^G$.

Remark 4.14. Several aspects of the proof of the above Lemma will be used in the proof of the expanded version of Theorem 4.15. Let $D = \sum a_i \Gamma_i$ be the decomposition of D into mutually distinct, prime components. Let $H_k^{(i)} \in |mM - \Gamma_i|$ where M is a divisor chosen so that $H_k^{(i)}$ is very ample for all i. Let m be such that mD is integral.

- $K_Y = \gamma^*(K_X) + (m-1)\left(\sum(\gamma^*\Gamma_i)_{red} + \sum(\gamma^* H_k^{(i)})_{red}\right)$
- $\gamma^*\Gamma_i = m\left((\gamma^*\Gamma_i)_{red}\right)$

Finally, applying Theorem 4.12 and Lemma 4.13 to Tanaka's proof gives the following:

Theorem 4.15. *[Tan15c, 2.6] Let X be a smooth projective surface over an algebraically closed field of characteristic $p \geq 0$. Let N be a nef and big $\mathbb{R}$-cartier and B a nef and big $\mathbb{Q}$-divisor whose fractional part is simple normal crossing, whose fractional part has no denominators divisible by the characteristic. Then there exists an r, computable in terms of* *the intersection numbers of the components of* B, N, K_X*, and any one fixed ample divisor* A *on* X [1] *such that*

$$H^i\left(X, \mathcal{O}_X\left(K_X + \ulcorner B \urcorner + rN + N'\right)\right) = 0$$

for every $i > 0$*, and every nef* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* N' *such that* $rN + N'$ *is a* $\mathbb{Z}$*-divisor.*

Proof. (A slightly different proof than the cited Theorem). The fractional part of $B + rN + N'$ is equal to the fraction part of B when $rN + N'$ is a $\mathbb{Z}$-divisor. Thus we apply Lemma 4.13 to obtain a degree m cover $\gamma : Y \to X$ (independent of r and N') where Y is a smooth surface.

Now I claim that I can choose r depending only on the intersections of K_X, B and the degrees of components of $B = \sum a_i \Gamma_i$. such that Y is general type, K_Y is ample, and such that $(B + rN)^2 > K_Y^2$, so that Theorem 4.12 can be applied.

As B is big and nef, there exists an effective divisor D such that for all $j \gg 0$, $B - \frac{1}{j}D \equiv A_j$, with A_j ample. Applying [Laz04, 2.2.15, 2.2.19] we can even compute j in terms of the intersection numbers of A and B such that D is big and $\frac{1}{j}B.D > \frac{1}{j^2}D^2$.

Following the proof of [KMM87, 1-1-1], let $M = k_j A_j$ be very ample (we can compute k_j by [Ter99, DCF15]) and let m_j be such that m_j clears the denominators of the components of B and such that $m_j M - \Gamma_i$ is very ample for all i (again we can compute such an m_j by [Ter99, DCF15]: in the case that $\Gamma_i^2 < 0$, then ensure k_j is large enough so that $k_j A.\Gamma_i > -\Gamma_i^2$, take $\Gamma_i' = \Gamma_i + M$ and by [Ter99, DCF15] find (computable) m' such that $m'M - \Gamma_i' = m''M - \Gamma_i$ is very ample.) Replace m_j by m'' if $m'' > m$, and let $H_i \in |m_j M - \Gamma_i|$ for all i (thus $\frac{1}{m_j}(\Gamma_i + H_i) \sim M$ is very ample). Applying Remark 4.14 we then have

$$K_Y = \tau^* K_X + (m_j - 1)\left(\sum (\tau^*\Gamma_i)_{red} + \sum \left(\tau^* H_k^{(i)}\right)_{red}\right)$$

$$= \tau^* K_X + \tau^*\left(\sum_i \left(\Gamma_i + H_i - \frac{1}{m_j}(\Gamma_i + H_i)\right)\right)$$

$$= \tau^* K_X + \tau^*\left((m_j - 1) M\right)$$

so that Y is general type and if m_j is chosen large enough, K_Y is even nef. Thus,

$$vol(Y) = K_Y^2 = (K_X + (m_j - 1)k_j A)^2$$

---

[1] In practice, I will apply this statement in a family where the ample divisor A has the same intersection numbers on all the fibers.

so, to satisfy the hypothesis of Theorem 4.12, it suffices to pick r large enough that

$$(B + rN)^2 - 2 > (K_X + (m_j - 1)k_j A_j)^2$$

or, by construction of A, such that

$$(B + rN)^2 - 2 > (K_X + (m_j - 1)k_j B)^2$$

Finally,

$$H^1\left(K_X + \ulcorner B \urcorner + rN + N'\right)$$

$$= H^1\left(K_X + \ulcorner B + rN + N' \urcorner\right)$$

$$= H^1\left(\gamma_*\left(K_Y + \gamma^*\left(B + rN + N'\right)\right)\right)^G$$

$$= H^1\left(Y, K_Y + \gamma^* B + r\gamma^* N + \gamma^* N'\right)^G.$$

the last term vanishing by Theorem 4.12. □

Now I restate Tanaka's vanishing theorem, with a note on the computability in certain circumstances.

Theorem 4.16. *(KLT vanishing for normal surfaces c.f. [Tan15c, 2.11]) Let (X, Δ) be a normal projective Kawamata log-terminal surface over an algebraically closed field of characteristic $p \geq 2$, where $K_X + \Delta$ is an effective $\mathbb{R}$-divisor. Let N be a nef and big $\mathbb{R}$-cartier $\mathbb{R}$-divisor. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D - (K_X - \Delta)$ is nef and big. Then there exists a characteristic-free constant r_0* [2] *such that*

$$H^i\left(X, \mathcal{O}_X\left(D + rN + N'\right)\right) = 0$$

for every $i > 0$, every positive real number $r \geq r_0$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor N' such that $rN + N'$ is a Cartier divisor.

---

[2] By Theorem 4.15, except for in possibly a finite number of characteristics $p > 0$, r_0 is computable in terms of the number of components Δ and in terms of the self intersections of these components, and their multiplicities. Otherwise Tanaka's original statement gives just the existence of an r_0 depending only on the relevant divisors. In any case, if the family happens to be over $\mathbb{Z}$ or a similar Dedekind domain, then you can use Tanaka's original constant for the finite number of characteristics dividing denominators of the boundary divisor

Proof. The computability follows easily from [Tan15c, 2.11] except that Weak Kawamata Viehweg Vanishing in the proof is replaced by Theorem 4.15. Since, in this paper, I can apply this theorem when X is smooth and $\llcorner \Delta \lrcorner = 0$, here is the proof in that simple case:

$$H^1\left(D + rN + N'\right)$$

$$= H^1\left(K_X + D - K_X - \llcorner \Delta \lrcorner + rN + N'\right)$$

$$= H^1\left(K_X + D - K_X + \ulcorner -\Delta \urcorner + rN + N'\right)$$

$$= H^1\left(K_X + \ulcorner D - K_X - \Delta \urcorner + rN + N'\right)$$

since we can move the integral divisor $D - K_X$ into the round up. This last term vanishes by Theorem 4.15. The vanishing of H^2 follows easily using Serre Duality. Otherwise if X is normal, one can keep track of the extra components of Δ gained in blowing up, and compute from that. □

This gives the following version of Theorem 3.23:

Corollary 4.17. *Let (X, Δ) be a Kawamata log-terminal pair of relative dimension 2 over a Dedekind Domain R such that any residue field k is perfect of characteristic $p > 0$ and with perfect fraction field K. Assume that $K_X + \Delta$ is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists an m_0 such that for $m \gg 0, m \in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

4.5 Generalization of Terakawa's Basepoint Free Theorem

The goal of this section is to give a specialized version of Terekawa's basepoint-free theorem which applies to Kawamata log-terminal pairs. The following lemma is key to the proof.

Lemma 4.18. *[Ter99, 2.2] Let (X, Δ) a minimal simple normal crossings projective KLT pair of general type and dimension 2 defined over an algebraically closed field of characteristic $p \geq 2$. Let Z be a 0-cycle with deg $Z = d + 1$ where d is a nonnegative integer. Assume that L is a line bundle such that $K_X + \Delta + L$ is Cartier and $H^1\left(K_X + \Delta + L\right) = 0$. Assume that $K_X + L$ is $(d-1)$-very ample and the restriction map is not surjective:*

$$\Gamma\left(K_X + \Delta + L\right) \to \Gamma\left(\mathcal{O}_Z\left(K_X + \Delta + L\right)\right).$$

Then there exists a rank 2 locally free sheaf E on X which is given by the short exact sequence

$$0 \to \mathcal{O}_X \to E \to I_Z(L) \to 0$$

where I_Z is the ideal sheaf of Z.

Proof. Follows easily from cited theorem. ☐

Using this, we get a characteristic-free basepoint-free theorem.

Theorem 4.19. *(c.f. [Ter99]) Let (X, Δ) a minimal simple normal crossings projective KLT pair of general type and dimension 2 defined over an algebraically closed field of characteristic $p \geq 2$. Let $L = m(K_X + \Delta)$ be a nef and big line bundle on X. Assume that $l = L^2 \geq K_X^2 - 3\chi(\mathcal{O}_X) + 2 + 4d + 5$ for $d \geq 0$. Then there is a computable constant m_0 (depending only on the intersections K_X and Δ and the number of components of Δ) such that $mm_0(K_X + \Delta)$ is basepoint-free.*

Proof. Let m_0 be large enough so that the hypothesis of Theorem 4.16 holds i.e. such that

$$H^1(K_X + \Delta + L) = H^1(m_0(K_X + \Delta)) = 0.$$

If $D = K_X + \Delta + L$ is not base-point free, then there is a 0-cycle Z of degree 1 such that the restriction map

$$\Gamma(X, D) \to \Gamma(Z, D|_Z)$$

is not surjective. By Lemma 4.18, there exists a rank 2 locally free sheaf E on X which is given by the short exact sequence

$$0 \to \mathcal{O}_X \to E \to I_Z(L) \to 0$$

where I_Z is the ideal sheaf of Z. The rest of the argument follows exactly as in the cited Theorem. ☐

4.6 Finite Generation of Adjoint Rings

In this section, I use the techniques of [CL14] and apply the proof of Theorem 4.3 to prove finite generation of certain adjoint rings.

Definition 4.20. Let $(X, \sum S_i)$ a log smooth, log-canonical projective pair where X is a two dimensional, normal variety over an algebraically closed field of characteristic $p > 0$

with S_i distinct prime divisors, $V = \sum \mathbb{R} S_i \subset Div_{\mathbb{R}}(X)$. Define the following sets, the first of which is clearly a polytope.

$$\mathcal{L}(V) = \left\{ \sum_{i=1}^{n} a_i S_i \in V \mid a_i \in [0,1] \right\}$$

$$\mathcal{E}(V) = \{ \Delta \in \mathcal{L}(V) \mid |K_X + \Delta|_{\mathbb{R}} \neq \emptyset \}.$$

Given $f : X \to Y$ a birational contraction, let $\mathcal{C}_f(V)$ denote the closure in $\mathcal{L}(V)$ of

$$\{ \Delta \in \mathcal{E}(V) \mid f \text{ is a log terminal model of } (X, \Delta) \}.$$

C.f. [CL14, 2.13], (the statement is for characteristic 0 in any dimension by [SC11, 3.4], but in the surface case here merely relies on [Tan14, 0.2]) then there are birational contractions $f_i : X \to Y_i$ such that $\mathcal{C}_{f_i}(V), ..., \mathcal{C}_{f_k}(V)$ are rational polytopes and

$$\mathcal{E}(V) = \bigcup_{i=1}^{k} \mathcal{C}_{f_i}(V)$$

so that $\mathcal{E}(V)$ is also a polytope. Finally, for $B_1, ..., B_m \in \mathcal{E}(V)$, let $\mathcal{C} = \mathcal{C}_{B_1,...,B_m}$ be the rational polytope spanned by these B_i and define $\mathcal{C}_i = \mathcal{C} \cap \mathcal{C}_{f_i}(V)$.

Proposition 4.21. *Let X/R be an arithmetic threefold. Let $\{(X, \Delta_i)\}_{i \in \{1,...,k\}}$ be a big, log-smooth, $\mathbb{Q}$-Cartier Kawamata log-terminal pair for each i and such that $\sum_{i=1}^{k} \Delta_i$ has simple normal crossings support. Then there exists a constant m_0 such that on any fiber $X_r, r \in R$ the ring*

$$R(X_r, K_X + \Delta_1|_{X_r}, ..., K_X + \Delta_k|_{X_r})$$

is finitely generated in degree m_0. Furthermore, if all Δ_i are of the form $\Delta + A_i$ and some ample divisors A_i with $\|A_i - A_j\| \ll 1$, then

$$R(X, K_X + \Delta_1, ..., K_X + \Delta_k)$$

is finitely generated.

Proof. For the second statement, choosing $\|\Delta_i - \Delta_j\| \ll 1$ enforces that steps of the $K_X + \Delta_i$ minimal model program are steps of the $K_X + \Delta_j$ minimal model program. Thus, we can argue as in Theorem 4.3, arriving at a minimal model (after subtracting the non-nef part of Δ) for each pair, so the result follows by Theorem 4.10 and [HK00b, 2.8].

For the first statement, I give a proof based on [CL14, 2.18] which applies in the KLT case). For the first statement, replace X by an arbitrary fiber, and after base change, we may assume that fiber is algebraically closed. Set $\mathcal{C} = \mathcal{C}_{\Delta_1,\ldots,\Delta_k}$ and $\mathcal{C}_i$ as in definition 4.20. Let $\sum supp\,(\Delta_i) = \sum S_i$ for S_i distinct prime divisors, $D_i = q\,(K_X + \Delta_i)$ for some q making $K_X + \Delta_i$ Cartier, and $\mathcal{M} = \sum_{i=1}^{p} \mathbb{Z}D_i$. Let $\mathcal{M}^{(N)} = \sum_{i=1}^{p} \mathbb{Z}ND_i$. As in the cited theorem, for each i, there is a constant M' and generators of $\mathcal{M} \cap \mathbb{R}_+\,(K_X + \mathcal{C}_i)$ of the form $M'\,(K_X + B)$ for some $B \in \mathcal{L}\,(V)$. Let B_{ji} span each $\mathcal{C}_j$ and pick a sufficiently small multiple of an ample divisor A, such that, as in the proof of Theorem 4.5, (since the B_{ij} are by definition log-smooth) the adjoint rings

$$R(X, K_X + B_{ij}, K_X + B_{ij} + tA)$$

are all finitely generated. Applying [Tan14] and Theorem 4.19, there is a bounded constant $a \in \mathbb{N}$ (depending only on the intersections of the relevant divisors and not depending on the fiber) such that for all j, i, $L_{ji} := a\,(f_j)_*\,(K_X + B_{ji})$ is basepoint free. Thus, applying [HK00b, 2.8] the rings

$$R(X, \mathcal{C}_j \cap \mathcal{M}^{(M/q)})$$

are finitely generated, which gives an asymptotic bound for the first statement.

For a characteristic free bound, instead of applying [HK00b, 2.8], I apply the proof of [CZ14, 2.7] for each j as follows: Fix an i and set $L_i = L_{ji}$. Let $G = \sum_{i=1}^{k} b_i L_i$ for some integers $b_1, ..., b_k \geq 0$. If l is in $1, ..., k$, and $b_\ell > n + 2$ then we must show that

$$H^0(X, \mathcal{O}_X(G - L_\ell)) \otimes H^0(X, \mathcal{O}_X(L_\ell)) \to H^0(X, \mathcal{O}_X(G))$$

is surjective. Now I continue as in [CZ14, 2.7]. We skip the first part of the proof, since each given pair is already big. Now let $V = H^0(X, \mathcal{O}_X(L_\ell))$ and $\mathcal{V} = V \otimes \mathcal{O}_X$ with V having dimension R. Then

$$\mathcal{V} \otimes \mathcal{O}_X(-L_\ell) \to \mathcal{O}_X$$

is surjective. Twisting by appropriate line bundles gives

$$0 \to \wedge^{r+1}\mathcal{V} \otimes \mathcal{O}_X(G - (r-1)L_\ell) \to \cdots$$

$$\to \wedge^2\mathcal{V} \otimes \mathcal{O}_X(G - L_\ell) \to \mathcal{V} \otimes \mathcal{O}_X(G) \to \mathcal{O}_X(G + L_\ell) \to 0$$

As in the source, $G - jL$ is nef and big. Thus taking a suitably large a allows Theorem 4.16 to be applied so that for $j > 0$,

$$H^j(X, \mathcal{O}_X(\sum_{i \neq \ell} b_i L_i + (b_\ell - j)L_\ell)) = 0.$$

The rest of the finite generation on the fiber is exactly as in [CZ14, 2.7]. $\square$

CHAPTER 5

TERMINATION WITH SCALING

In [CL13], the authors noted that once finite generation of the adjoint rings is proven, then it is possible to prove termination of the Minimal Model program in characteristic 0. In characteristic 0, and in the mixed characteristic arithmetic threefold case (by Chapter 4) we only know that the general type adjoint rings are finitely generated, and thus, using merely these techniques, it is only possible to prove minimal models exist in those cases. (Luckily, in the arithmetic threefold case, there are some additional tricks, which I will use in chapter 6, to study pairs which are not of general type).

In this chapter, I use the techniques of [CL13] to prove termination in the general type case. Certain things do not always work exactly the same way, as the setting is mixed characteristic. In this chapter, R will denote a Dedekind domain with perfect residue fields.

5.1 Cone, Rationality, and Contraction

In Chapter 3, it was possible to run the minimal model program for arithmetic threefolds using the cone theorem on a fiber after the diminished base-locus was removed. In this section, I prove a cone theorem that works without that hypothesis, and instead uses the adjoint ring finite generation proven in Theorem 4.21.

Definition 5.1. [CL13, def 4.1] Let W a finite dimensional real vector space, $C \subset W$ a closed convex cone spanning W, and $v \in W$. The visible boundary of C from v is

$$V = \{w \in \partial C \mid [v, w] \cap C = \{w\}\}.$$

Corollary 5.2. *(c.f. [CL13, 4.2]) (Kawamata's Rationality, Cone and Contraction Theorem) Let (X, Δ) be a pseudo-effective pair of relative dimension 2, proper over R. Assume that $v_0 = [K_X + \Delta]$ is $\mathbb{Q}$-Cartier and is obtained from the log smooth Kawamata log-terminal pair $\phi : (X', \Delta') \to (X, \Delta)$ after a finite number of steps of the minimal model program with scaling associated to a big divisor A with $K_X + \Delta + tA$ nef on X/R for some $t > 0$. Let V be the visible boundary of $Nef(X/R)$ from $v_0 \in N^1(X)_{\mathbb{R}}$ and*

$u_0 = [K_X + \Delta + t_0 A] \in V \cap (v_0, [K_X + \Delta + tA])$. Then there exists a locally polyhedral neighborhood $U \subset V$ of u_0 such that $\forall u' \in U$, u' is semiample.

Proof. This is essentially a local version of the cited theorem. Let $\mathcal{C} = \mathbb{R}_+ v_0 + Nef\,(X/R)$. Let u_0' be a rational point of $N^1(X/R)$ sufficiently close to u_0 such that $u_0 \in B(u_0', \epsilon)$ is a sup-norm ball with rational vertices $w_j = [K_X + \Delta + A_j]\,,\ j = 1, ..., r$ for $\|A_j - t_0 A\| \ll 1$ which are Kawamata log-terminal and such that there exist big, log-smooth, Kawamata log-terminal pairs $K_{X'} + \Delta' + A_j'$ with $\phi_*\left(K_{X'} + \Delta' + A_j'\right) = K_X + \Delta + A_j$ on X' , and such that ϵ is small enough that the w_j pull back to divisors having log smooth support on X'. Let $\mathcal{B}$ denote the convex hull of $B(u_0', \epsilon)$. For ϵ sufficiently small, steps of the $K_{X'} + \Delta' + t_0 A$ minimal model program are also steps of the $K_X + \Delta + A_j'$ minimal model program. Therefore there is an isomorphism of rings

$$\mathfrak{R} := R\left(X, K_X + \Delta + A_1, ..., K_X + \Delta + A_r\right)$$

$$\approx R\left(X', K_{X'} + \Delta' + A_1', ..., K_{X'} + \Delta' + A_r'\right)$$

so that by Theorem 4.21, these rings are finitely generated. Let

$$\varphi : \sum \mathbb{R}_+ \left(K_X + \Delta + A_j\right) \to N^1(X)_{\mathbb{R}}$$

the natural projection so that

$$\varphi\left(\sum \mathbb{R}_+ \left(K_X + \Delta + A_j\right)\right) = \mathbb{R}_+ \mathcal{B}.$$

Theorem 4.8 implies that $Supp\ \mathfrak{R}$ is a rational polyhedral cone which intersects the interior of the nef cone. Therefore, by Theorem 4.8(2), if U is the portion of $\partial Nef\,(X/R)$ contained in $B(u_0', \epsilon)$, then $\varphi^{-1}(U) \subset Supp\,(\mathfrak{R})$. Let $Supp\ \mathfrak{R} = \bigcup \mathcal{L}_k$ be the coarsest subdivision given by Theorem 4.8(3). For some k, $\mathcal{L}_k \cap \varphi^{-1}(Amp\ X) \neq \emptyset$, so by Theorem 4.11, $\mathcal{L}_k = Supp\ \mathfrak{R} \cap \varphi^{-1}(Nef\,(X))$, and thus U is locally polyhedral (being contained in $\partial \mathcal{L}_k$).

Finally, If $u' \in U$, with divisor D then there exists an ample $\mathbb{Q}$-divisor A' such that $D \sim_{\mathbb{Q}} K_X + \Delta + A'$. By choice of ϵ, there is a log smooth divisor D' on X' with $\phi_* D' = D$, and so that $R(K_X + \Delta + A', K_X + \Delta + A' + A'')$ and $R(K_X + \Delta + A')$ are finitely generated for any ample A'' such that $D + A'' \in B(u_0', \epsilon)$. Thus, Theorem 4.10 gives that u' is semiample. □

5.2 Existence of Flips

In this section, I recall several results from [CL13] which will be used in the proof of the termination of flips in the general type case, and then prove the existence of flips.

Lemma 5.3. *[CL13, Lemma 5.1]Let X and Y be $\mathbb{Q}$-factorial projective schemes of relative dimension 2, smooth over R. Let $f : X \to Y$ be a birational contraction and let $\tilde{f} : \mathbb{K}(Y) \approx \mathbb{K}(X)$ the induced isomorphism on function fields. Then:*

(1) $f_* div_X \varphi = div_Y f(\varphi)$ for every $\varphi \in \mathbb{K}(X)$;

(2) for every geometric valuation Γ on $\mathbb{K}(X)$, and for every $\varphi \in \mathbb{K}(X)$, we have $mult_\Gamma(div_X \varphi) = mult_\Gamma\left(div_Y \tilde{f}(\varphi)\right)$;

(3) if f is an isomorphism in codimension one, then $f_* : Div_{\mathbb{R}}(X) \to Div_{\mathbb{R}}(Y)$ is an isomorphism, and for every $D \in Div_{\mathbb{R}}(X)$, there is an isomorphism $H^0(X, D) \approx H^0(Y, f_* D)$.

Proof. Follows easily from the cited Theorem. ☐

Lemma 5.4. *[CL13, 5.2] Let X and Y be proper of relative dimension 2, over R and $f : X \to Y$ a birational map which is an isomorphism in codimension one. Let $\mathcal{C} \subset Div_{\mathbb{R}}^{eff}(X)$ be a cone, and fix a geometric valuation Γ of X. Then the asymptotic order of vanishing σ_Γ is linear on $\mathcal{C}$ if and only if it is linear on $f_*\mathcal{C} \subset Div_{\mathbb{R}}^{eff}(Y)$.*

Proof. As in the cited theorem, substituting Lemma 5.3 as necessary. ☐

Lemma 5.5. *[HK00a, 1.7] Let $f : X \to Y$ and $g : X \to Z$ be birational contractions where all spaces considered are schemes of relative dimension 2 over R. Suppose $f^*(A) + E = g^*(B) + F$ for horizontal Cartier divisors A (which is ample), B (which is nef), E f-exceptional, and F g-exceptional divisors extending to both fibers. Then $f \circ g^{-1} : Z \to Y$ is regular.*

Proof. It suffices to apply the negativity lemma (the usual proof holds c.f. [KM98, 3.39]) so that $E = F$ and thus $f^*(A) = g^*(B)$. The result then follows easily. ☐

Lemma 5.6. *[CL13, 6.4] Let (X, Δ) a projective Kawamata log-terminal pair, and $f : X \to Y$ a composite of $(K_X + \Delta)$-divisorial contractions and $(K_X + \Delta)$-flips. Then for every resolution*

$$\begin{tikzcd}
W \arrow[r, "p"] \arrow[d, "q"'] & X \\
Y \arrow[ur, "f"'] &
\end{tikzcd}$$

of f,

$$p^*(K_X + \Delta) = q^*(K_Y + f_*\Delta) + E$$

with $E \underset{\neq}{>} 0$ *a* q*-exceptional divisor. Therefore* f *cannot be an isomorphism. By the above formula,*

$$H^0(X, K_X + \Delta) \approx H^0(Y, K_Y + f_*\Delta).$$

Proof. Follows easily from [KM98, 3.38]. □

Lemma 5.7. *[Har77, Exc II.4.2] Let* X *be a reduced scheme,* Y *a separated scheme, and let* f *and* g *be two morphisms from* X *to* Y*. Assume that* $f|_U = g|_U$ *on a Zariski dense open subset* $U \subset X$*. Then* $f = g$*.*

Finally the existence of flips.

Theorem 5.8. *Let* (X, Δ) *be a projective* $\mathbb{Q}$*-factorial Kawamata log-terminal big pair of relative dimension* 2 *over* R*. Suppose that* $f : X \to Y$ *is a* $K + \Delta$ *flipping contraction. If* $R(K_X + \Delta)$ *is finitely generated, then the* $(K + \Delta)$*-flip of* f *exists.*

Proof. Follows easily as in [KM98, 6.2], [CL13, 6.3]. As in [Kaw94, 5.3], existence of the flip is equivalent to surjectivity of

$$f_*\mathcal{O}_X(m_0(K_X + \Delta)) \otimes f_*\mathcal{O}_X(nm_0(K_X + \Delta)) \to f_*\mathcal{O}_X((n+1)m_0(K_X + \Delta))$$

which is stable under flat base change. Thus, it is safe to assume that R is a complete discrete valuation ring with algebraically closed residue field. As in definition 4.1 the above surjectivity is equivalent to finite generation of the relative adjoint ring

$$\mathfrak{R}_f := \bigoplus_{n \geq 0} (f_R)_*\mathcal{O}_X(n(K_X + \Delta))$$

The above ring is finitely generated if there is an affine open covering of Z by sets $U_i = Spec\ A_i$ such that $\mathfrak{R}_f|_{U_i}$ is the sheaf of a finitely generated A_i algebra. Thus we may assume that Z is affine. The theorem follows, since if $R(K_X + \Delta)$ is finitely generated, then it is finitely generated over an affine set. □

Remark 5.9. By [EKW04, 1.2], the flip of f in the situation above is $\mathbb{Q}-$factorial.

5.3 Termination with Scaling

In this section, I show that the minimal model program with scaling can be run for arithmetic threefolds of general type in an analogous manner to definition 2.13

Theorem 5.10. *Let* (X, Δ) *be a projective* $\mathbb{Q}$*-factorial Kawamata log-terminal pair of relative dimension* 2 *over* R*. Suppose* $K_X + \Delta$ *is big and log smooth over* R*. Then the* *minimal model program for (X, Δ) with scaling of an ample divisor A can be run, resulting in a terminating sequence of flips and divisorial contractions.*

Proof. I begin similar to [CL13, 6.2]. Let A be a big divisor such that $K_X + \Delta + A$ is nef. Let α_1 be the smallest positive real number such that $K_X + \Delta + \alpha_1 A$ is nef. Denote by $\varphi : Div_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection, and let $\| \cdot \|$ any norm on $N^1(X)_{\mathbb{R}}$. Pick finitely many big $\mathbb{Q}$-divisors $H^1, ..., H^r$ (for example perturbations of $\Delta + \alpha_1 A$) such that:

(1) $\|\varphi(\Delta + \alpha_1 A) - \varphi(H^i)\| \ll 1$ for all i,

(2) writing $\mathcal{C} = \sum_{i=1}^{r} \mathbb{R}_+ (K_X + H^i) \subset Div_{\mathbb{R}}(X)$, we have $K_X + \Delta + \alpha_1 A \in int\ \mathcal{C}$, and the dimension of the cone $\varphi(\mathcal{C}) \subset N^1(X)_{\mathbb{R}}$ is $dim\ N^1(X)_{\mathbb{R}}$,

(3) (X, H^i) is Kawamata log-terminal and log smooth for all i with $supp\left(K_X + \sum H^i\right)$ simple normal crossings.

In the first step, since H^i are log smooth, then by Theorem 4.21, the ring

$$R := R\left(X, K_X + \Delta, K_X + H^1, ..., K_X + H^r\right)$$

is finitely generated. Thus applying Theorem 4.8, there exists a rational polyhedral subdivision $Supp\ R = \bigcup C^j$ and by Theorem 5.2 there is a rational codimension 1 face of $Nef(X)$. Thus $\alpha_1 \in \mathbb{Q}_+$, and we can find an extremal ray $\mathfrak{R} \subset \overline{NE}(X)$ dual to this face which will satisfy $(K_X + \Delta + \alpha_1 A).\mathfrak{R} = 0$ and $(K_X + \Delta).\mathfrak{R} < 0$. By Theorem 5.2, this ray can be contracted under a birational contraction: $f' : X \to X'$.

If f' is a divisorial contraction, write $X_2 = X'$, $f_1 = f'$ and set $\Delta_2 = f_{1*}\Delta$, $A_2 = f_{1*}A$, so A_2 is again big and $K_{X_2} + \Delta_2 + \alpha_1 A_2$ is nef. By properties 1 and 3 above, setting $H_2^r = f_{1*}H^r$, we have finite generation of $R(X_2, K_{X_2} + \Delta_2)$, $R(X_2, K_{X_2} + \Delta_2, K_{X_2} + \Delta_2 + A_2)$, and

$$R_2 := R\left(X_2, K_{X_2} + \Delta_2, K_{X_2} + H_2^1, ..., K_{X_2} + H_2^r\right)$$

$$\approx R\left(X, K_X + \Delta, K_X + H^1, ..., K_X + H^r\right)$$

so the argument in the previous paragraph gives another extremal ray and the process can be repeated.

If, on the other hand, f' is a small contraction, then by Theorem 5.8, and applying Theorem 4.3, the flip of the contraction exists (c.f. definition 2.18). Let X_2 be the flip, write f_1 for the induced morphism to the flip, and set $\Delta_2 = f_{1*}\Delta$, $A_2 = f_{1*}A$, so A_2 is again big and $K_{X_2} + \Delta_2 + \alpha_1 A_2$ is nef. By Lemma 5.3, we have finite generation of

$$R(X_2, K_{X_2} + \Delta_2),$$

$$R(X_2, K_{X_2} + \Delta_2, K_{X_2} + \Delta_2 + A_2),$$

and we can again set $H_2^r = f_{1*}H^r$, so that

$$R_2 := R\left(X_2, K_{X_2} + \Delta_2, K_{X_2} + H_2^1, ..., K_{X_2} + H_2^r\right)$$

is also finitely generated. Thus we can similarly repeat the process.

Now suppose there is an infinite sequence of flips starting at step j. I will show leads to a contradiction. I proceed almost verbatim as in [CL13, 6.5]. By construction, C^j contains an open neighborhood of the nef divisor $K_{X_j} + \Delta_j + \alpha_j A_j$, so C^j contains ample divisors in its interior. Thus the cone $\varphi\left(Supp\ R_j\right) \subset N^1\left(X_j\right)$ also has dimension $dim\ N^1\left(X_j\right)_{\mathbb{R}}$. Let $Supp\ R_j = \bigcup_k C_k^j$ the coarsest finite rational polyhedral subdivision from Theorem 4.8(3). For $i > j$, let $C_k^i \subset Div_{\mathbb{R}}\left(X_i\right)$ denote the proper transform of C_k^j and $C^i \subset Div_{\mathbb{R}}\left(X_i\right)$, the proper transform of C^j. By Lemma 5.4, for every geometric valuation Γ, the asymptotic order function σ_Γ is linear on each C_k^j.

By construction if $0 < \alpha \leq \alpha_1$, then $K_{X_j} + \Delta_j + \alpha A_j \in Int\ C^j$, so $K_{X_i} + \Delta_i + \alpha_i A_i \in int\ C^i$ for all $i > j$. Since $K_{X_i} + \Delta_i + \alpha_i A_i$ is nef, then by requirement (4) above and Lemma 4.9, $K_{X_i} + \Delta_i + \alpha_i A_i \in Supp\ R_i$. Thus, by Theorem 4.11, for each i there exists an index k such that the image of C_k^i in $N^1\left(X_i\right)_{\mathbb{R}}$ is a subset of Nef(X_i). Therefore,

$$\varphi\left(C_k^1\right) \subset \left(f_{i-1} \circ \cdots \circ f_1\right)^* Nef\left(X_i\right).$$

Since there are finitely many cones C_k^j, there are two indices p and q such that

$$\left(f_{p-1} \circ \cdots \circ f_1\right)^* Nef\left(X_p\right)$$

and

$$\left(f_{q-1} \circ \cdots \circ f_1\right)^* Nef\left(X_q\right)$$

share a common interior point. Thus, by Lemma 5.5, the map $X_p \to X_q$ is a morphism. By Lemma 5.7, it is therefore an isomorphism. This contradicts Remark 5.6. □

CHAPTER 6

MINIMAL MODELS FOR ARITHMETIC THREEFOLDS

In this chapter, R will be a Dedekind domain with perfect residue fields, and I will show the existence of Log Minimal Models in the non- general type case.

6.1 Reductions

First I define a "weak log-canonical model" which is sort of an intermediate step to the minimal model.

Definition 6.1. Let (X, Δ) be a log-smooth arithmetic threefold pair with $K_X + \Delta$ $\mathbb{Q}$-Cartier. A **weak log-canonical model** of (X, Δ) consists of a $\mathbb{Q}$-Cartier pair (X^W, Δ^W) and a birational contraction $\phi : (X, \Delta) \to (X^W, \Delta^W = \phi_* \Delta)$ such that for every ϕ-exceptional divisor $E \subset X$ the log-discrepancies satisfy:

$$a(E, X, \Delta) \leq a(E, X^W, \Delta^W)$$

This concept is useful since it allows us, as in chapter 3.23 to remove some of the non-nef parts of a given boundary divisor.

Lemma 6.2. *[HMX14, 2.8.3] Let (X, Δ) be a log smooth pair which is a scheme of dimension at most 3, with the coefficients of Δ belonging to $(0, 1]$, and with X projective. If (X, Δ) has a weak log canonical model, then there is a sequence $\pi : Y \to X$ of smooth blow ups of the strata of Δ such that if we write $K_Y + \Gamma = \pi^* (K_X + \Delta) + E$, where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_* \Gamma = \Delta$ and $\pi_* E = 0$ and if we write*

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma (Y, K_Y + \Gamma),$$

then $\mathbb{B}_- (Y, K_Y + \Gamma')$ contains no strata of Γ'. If Δ is a $\mathbb{Q}$-divisor, then Γ' is a $\mathbb{Q}$-divisor.

Proof. Follows easily from the cited theorem. ☐

Since it is sometimes useful, as in the proof of Theorem 3.23, to run the minimal model program simultaneously on the total space and the fiber, I note that by the invariance of plurigenera, the N_σ restricts nicely to a fiber:

Lemma 6.3. *Let (X, Δ) be a big, log-smooth Kawamata log-terminal pair of relative dimension 2 over a Dedekind Domain R. Assume that $K_X + \Delta$ is pseudo-effective and $\mathbb{Q}$-Cartier. Then for every $r \in R$*

$$N_\sigma \left(X, K_X + \Delta\right) |X_r = N_\sigma \left(X_r, K_{X_r} + \Delta_r\right).$$

Proof. As in [HMX14, 2.3.1], it suffices to pick an ample A and show that there exists an m_0 such that for all m we have

$$f_* \mathcal{O}_X \left(mm_0 \left(K_X + \Delta\right) + A\right) \twoheadrightarrow H^0 \left(X_r, \mathcal{O}_{X_r} \left(mm_0 \left(K_X + \Delta\right) + A\right)\right)$$

for $m \gg 0$. As in [Kaw94, 5.8], we may assume that R is a complete discrete valuation ring whose residue field is algebraically closed. Applying Theorem 3.14 and Theorem 3.23, it holds that

$$H^0 \left(X_r, \mathcal{O}_{X_r} \left(mm_0 \left(K_X + \Delta\right) + A\right)\right)$$

$$\approx H^0 \left(X, \mathcal{O}_X \left(mm_0 \left(K_X + \Delta\right) + A\right)\right) \otimes k\left(r\right)$$

with $k\left(r\right) \approx R/\mathfrak{m}R$, and so the lemma follows. □

Lemma 6.4. *[HMX14, 5.1] Let (X, Δ) be a log-smooth Kawamata log-terminal pair and (X, Φ) a divisorially log terminal pair, both of relative dimension two over R. Let*

$$\Delta(t) = (1 - t)\Delta + t\Phi$$

Suppose that $f : X \dashrightarrow Y$ is a step of the $(K_X + \Delta(t))$-Minimal Model program over R and set $\Gamma = f_*\Delta$. If $K_{X_k} + \Delta_k$ is nef, then there exists $\epsilon > 0$ so that f is $(K_X + \Delta)$-trivial in a neighborhood of X_k whenever $0 < t < \epsilon$.

Proof. Suppose that C is an extremal ray corresponding to f. If $(K_X + \Delta).C > 0$, then C is a $(K_X + \Phi)$-negative extremal ray. The existence of the contraction corresponding to this ray is preserved by base change (c.f. [Kaw94, 2.3]), so we may assume that R is a complete discrete valuation ring with algebraically closed residue field. Looking on the special fiber, the length of the contracted extremal ray is bounded, so following the inequalities in [HMX14, 5.1], gives a contradiction for some $0 < t < \epsilon$. □

Lemma 6.5. *[HMX14, 5.3] Let X/R a projective scheme of relative dimension 2 over R. Let (X,Δ) be a log smooth divisorially log-terminal pair with X both $\mathbb{Q}$-factorial and projective and Δ a $\mathbb{Q}$-divisor. If Φ is a $\mathbb{Q}$-divisor such that*

$$0 \le \Delta - \Phi \le N_\sigma\left(X, K_X + \Delta\right),$$

then steps of the $K_X + \Phi$-minimal model program are steps of the $K_X + \Phi + t\left(\Delta - \Phi\right)$ minimal model program when t is sufficiently small. Furthermore, termination for (X,Φ) implies termination for (X,Δ).

Proof. (Similar proof to the cited Theorem). Let $f : X \to Y$ the result of running a $(K_X + \Phi)$ minimal model program and let $Y \to Z$ be the ample model of $K_X + \Phi$. Let $\Delta_t = t\Delta + (1-t)\Phi$ so that if $0 < t \ll 1$, then f is also a $(K_X + \Delta_t)$-MMP by Lemma 6.4. Thus every step of the $K_X + \Delta_t$ with scaling of an ample divisor is $(K_X + \Phi)$-trivial, so after finitely many steps there is a model $g : X \to W$ contracting the components of $N_\sigma\left(X, K_X + \Delta_t\right)$ (this holds with no changes from [HMX14, 2.7.1]). Thus, as $\mathit{supp}\left(\Delta - \Phi\right) \subset \mathit{supp}\ N_\sigma\left(X, K_X + \Delta\right) = \mathit{supp}\ N_\sigma\left(X, K_X + \Delta_t\right)$, then $g_*\left(K_Y + \Phi\right)$ is nef (alternatively nef and semiample) whenever $g_*\left(K_X + \Delta\right)$ is, and [HMX14, 2.7.2] (which holds using only the log smooth resolution) implies that g is a minimal model of (X,Δ), which is good when (X,Φ) has a good minimal model. ☐

Lemma 6.6. *[GL13, 2.3] Let X/R a scheme which is projective of relative dimension 2 over either R or over a field. Let (X,Δ) be a log smooth Kawamata log-terminal pair and $\varphi : W \to X$ a log resolution of (X,Δ). Choose Δ_W so that $K_W + \Delta_W = \varphi^*\left(K_X + \Delta\right) + E$ with Δ_W and E effective $\mathbb{Q}$-Weil divisors with no common component. Let I index the set of φ-exceptional prime divisors, and let*

$$F = \sum_{i \in I} F_i.$$

Setting $\Delta_W^\epsilon = \Delta_W + \epsilon F$, then (W, Δ_W^ϵ) is an ϵ-log smooth model of (X,Δ). Then for $0 < \epsilon \ll 1$, a (good) minimal model $\phi : (W, \Delta_W^\epsilon) \to \left(W_{min}, \Delta_{W_{min}}^\epsilon\right)$ is also a (good) minimal model for (X,Δ).

Proof. Follows easily from [BCHM10, 3.6.10, 3.6.11]. Also see [GL13]. ☐

6.2 Existence of Minimal Models Special Case

Lemma 6.7. *[HMX14, 3.1] Let (X,Δ) be a $\mathbb{Q}$-factorial Kawamata log-terminal pair of relative dimension 2 over R. Let k denote a residue field. Assume that*

$$\mathbb{B}_-\left(X_k, K_{X_k} + \Delta_k\right)$$

contains no non-canonical centres of (X_k, Δ_k). *Let* $f : X \to Y$ *be a step of the* $(K_X + \Delta)$-*MMP. If* f *is birational and* V *is a non-canonical centre of* (X, Δ), *then* V *is not contained in the indeterminacy locus of* f, V_0 *is not contained in the indeterminacy locus of* f_0, *and the induced maps* $\phi : V \to W$ *and* $\phi_k : V_k \to W_k$ *are birational, where* $W = f(V)$. *Let* $\Gamma = f_* \Delta$. *Then* $\mathbb{B}_-\left(Y_0, K_{Y_0} + \Gamma_0\right)$ *contains no non-canonical centres of* (Y_0, Γ_0) *(so we may repeat the process), and if* V *is a non-Kawamata log-terminal centre or* $V = X$, *then* $\phi : V \to W$ *and* $\phi_0 : V_0 \to W_0$ *are birational contractions. If* f *is a Mori fibre space, then* f_0 *is not birational.*

Proof. (Follows easily from the cited Theorem). Suppose f is birational. Let V be a non-canonical centre of (X, Δ). Let $g : X \to Z$ be the contraction of the extremal ray associated to f (so that $f = g$ unless f is a flip). Let $Q = g(V)$, and let $\psi : V \to Q$, be the induced morphism. As every component of V_k is a non-canonical centre of (X_k, Δ_k), then by hypothesis, components of V_k are not contained in $\mathbb{B}_-\left(X_k, K_{X_k} + \Delta_k\right)$, thus ψ_k is defined at each such component, hence is birational. By upper-semicontinuity of the fibers of ψ, ψ is birational, and thus $\phi : V \to Q \to W$, and ϕ_k are both birational.

Now suppose V is a non-Kawamata log-terminal centre or $V = X$, the above holds in the first case, as non-Kawamata log-terminal centres are non-canonical. Comparing the discrepancies of the differents of adjunction for Δ_k and Γ_k as in the cited proof shows that ϕ_k, f_k, and thus ϕ are birational contractions. On the other hand, if f is a Mori fibre space, then, as the dimension of the fibers of $f : X \to Y$ are upper-semicontinuous, f_k is not birational. □

Theorem 6.8. *[HMX14, 3.2] Let* (X, Δ) *be a* $\mathbb{Q}$-*factorial Kawamata log-terminal pair of relative dimension* 2 *over* R. *Assume that* $\mathbb{B}_-\left(X_k, K_{X_k} + \Delta_k\right)$ *contains no non-canonical centres of* (X_k, Δ_k) *and is log smooth. Let* $f : X_i \to Y := X_{i+1}$ *be a step of the* $(K_X + \Delta)$-*MMP. Then the minimal model program with scaling terminates using only divisorial contractions.*

Proof. (Similar proof to the cited theorem). Let $f : X \to Y$ the $(K_X + \Delta)$-MMP with scaling of an ample divisor A. Let $\Gamma = f_* \Delta$ and $B = f_* A$. By construction, $K_Y + tB + \Gamma$ is nef for some $t > 0$. By Lemma 6.7, $f : X \to Y$ is a birational contraction and $f_k : X_k \to Y_k$ is a birational contraction from $(X_k, tA_k + \Delta_k)$. If $K_X + \Delta$ is not pseudo-effective, then for $t > 0$, the result of f is a Mori fiber space, and by Lemma 6.7, Y_k is covered by

$K_{Y_k} + tB_k + \Gamma_k$-negative curves, contradicting bigness of $K_{X_k} + tA_k + \Delta_k$. Thus $K_X + \Delta$ is pseudo-effective, and given any $\epsilon > 0$, we may run the MMP until $t < \epsilon$. Now we conclude by Theorem 4.4. Letting ϵ be the constant given in that theorem, the minimal models for $(X_k, tA_k + \Delta_k)$ are all isomorphic for $t \in [0, \epsilon]$. Thus, once $t < \epsilon$, any additional step in the minimal model program with scaling must be an isomorphism on the special fiber, and thus an isomorphism. Thus there exists a minimal model (Y, Γ) for (X, Δ). □

6.3 Existence of Minimal Models General Case

Finally the main theorem of the thesis:

Theorem 6.9. *Let (X, Δ) be a Kawamata log-terminal pair of relative dimension 2, proper over R. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, pseudo-effective, and log smooth over R. Then the minimal model of (X, Δ) exists.*

Proof. I again start by repeating the reduction of [HMX14, 6.1]. Let $f_0 : Y_0 \to X_0$ be the birational morphism of Lemma 6.2. Under the log smooth hypothesis, and since the strata of Δ have irreducible fibers, and f_0 blows up strata of Δ_0, we may extend f_0 to a birational morphism $f : Y \to X/R$ which is a composition of smooth blow ups of strata of Δ. Write

$$K_Y + \Gamma = f^* (K_X + \Delta) + E$$

with $\Gamma \geq 0$ and $E \geq 0$, $f_*\Gamma = \Delta$, and $f_*E = 0$. Then (Y, Γ) is log smooth and the fibres of components of Γ are irreducible. By Lemma 6.6, (X, Δ) has a minimal model if (Y, Γ_ϵ) has a good minimal model, where $(Y, \Gamma_\epsilon) = (Y, \Gamma + \epsilon F)$ is the ϵ-log smooth model of (X, Δ) with F the sum of f-exceptional divisors and $0 < \epsilon \ll 1$.

Replace (X, Δ) by (Y, Γ_ϵ) and set $\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma (X_k, K_{X_k} + \Delta_k)$ so that

$$\mathbb{B}_- (X_k, K_{X_k} + \Theta_k)$$

contains no strata of Θ_k. Let $0 \leq \Theta \leq \Delta$ be the unique divisor such that $\Theta_k = \Theta|_{X_k}$. Let H relatively ample such that $(X, \Theta + H)$ is log smooth over R, and $K_X + \Theta + H$ is big. Thus there is a commutative diagram:

$$\begin{tikzcd}
\pi_*\mathcal{O}_X (m (K_X + \Theta) + H) \arrow[r] \arrow[d] & \pi_*\mathcal{O}_X (m (K_X + \Delta) + H) \arrow[d] \\
H^0 (X_k, \mathcal{O}_{X_k} (m (K_{X_k} + \Theta_k) + H_k)) \arrow[r] & H^0 (X_k, \mathcal{O}_{X_k} (m (K_{X_k} + \Delta_k) + H_k))
\end{tikzcd}$$

with surjective columns by Theorem 3.23 and [Liu02, 5.3.20(b)], and with the bottom row an isomorphism. Applying Nakayama's lemma gives an isomorphism on the top row, so that $\Theta \geq \Delta - \Delta \wedge N_\sigma\left(X, K_X + \Delta\right)$. Again applying Theorem 3.23, gives $\Theta = \Delta - \Delta \wedge N_\sigma\left(X, K_X + \Delta\right)$. Thus $\Delta - \Theta \leq N_\sigma\left(X, K_X + \Delta\right)$, so by Lemma 6.5, it suffices to find a minimal model for $\left(X, \Theta\right)$. Replacing (X, Δ) by (X, Θ), it suffices to assume that

$$\mathbb{B}_-\left(X_k, K_{X_k} + \Delta_k\right)$$

contains no strata of Δ_0. Letting A be an ample divisor, we run the minimal model program with scaling of A. Now the assumptions of Theorem 6.8 apply, so we know that (X, Θ) has a minimal model. $\square$

APPENDIX

SUPPLEMENTARY ALGEBRA RESULTS

In this appendix, I discuss some basic results on algebra which may be useful to understand the underlying objects discussed in the thesis.

A.1 Discrete Valuation Rings

This appendix contains some basic results on discrete valuation rings useful for understanding Lemma 3.6 in Chapter 3.

Definition A.1. A discrete valuation ring (discrete valuation ring for short) R is an integral domain which is an integrally closed noetherian local ring with Krull dimension one.

Lemma A.2. *Let A be a discrete valuation ring with fraction field K. Let L/K be any finite separable extension, and let B denote the integral closure of A in L. Then B is a finite A-module, and if A is complete, then so is B, and B is a discrete valuation ring. That is, finite extensions are complete and unsplit.*

Definition A.3. Let A, B be Noetherian local rings with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$. A local homomorphism $A \to B$ is said to be an unramified homomorphism of local rings if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,

(2) $\kappa(\mathfrak{m}_B)$ is a finite separable extension of $\kappa(\mathfrak{m}_A)$, and

(3) B is essentially of finite type over A (i.e. B is the localization of a finite type A-algebra at a prime.

Definition A.4. Let A, B be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be an étale homomorphism of local rings if it is a flat and unramified homomorphism of local rings. If Y is a locally Noetherian scheme, and $f : X \to Y$ is a morphism of schemes which is locally of finite type, then f is said to be étale if it is étale at all its points.

Lemma A.5. *[DERU13, 6.14] Let A be a noetherian integrally closed local ring with fraction field K and set $S = Spec(A)$. Let $\phi : X \to S$ be an étale cover. Then X is* *also normal, and in particular, it can be written as the coproduct of its (finitely many) irreducible components. Furthermore, given a connected component X_0 of X, the induced étale cover $X_0 \to S$ is the normalization of S in $k(S) \hookrightarrow k(X_0)$.*

From the above lemmas, it is clear that an étale cover of a complete discrete valuation ring results in a complete discrete valuation ring.

A.2 Riemann-Roch for Perfect Fields

The standard Riemann-Roch theorem for algebraic surfaces is stated over an algebraically closed field. This section originally appeared in my blog posts [Egb15, SurfDIOP2.0+], circa May 2015, since I was unable to find a reference for the material, however these facts are more recently stated in [BCZ15, 2.3].

Lemma A.6. *[Liu02, 7.3.16] Let X be a projective variety over a field k. Let*

$$0 \to F \to G \to H \to 0$$

be an exact sequence of coherent sheaves on X. Then

$$\chi(G) = \chi(F) + \chi(H).$$

Theorem A.7. *[Liu02, 7.3.17] Let X be a projective curve over a field k. Let D be a Cartier divisor on X. Then we have*

$$\chi(\mathcal{O}_X(D)) = deg\ D + \chi(\mathcal{O}_X).$$

Definition A.8. [Liu02, 4.1.2, 8.3.1]A normal Noetherian integral domain of dimension 0 or 1 is called a Dedekind domain. A normal locally Noetherian scheme of dimension 0 or 1 is called a Dedekind scheme. Let S be a Dedekind Scheme. We call an integral, projective, flat S-scheme $\pi : X \to S$ of dimension 2 a fibered surface over S. If $dim\ S = 0$ then X is an integral, projective, algebraic surface over a field. An irreducible Weil divisor D is called horizontal if $dim\ S = 1$ and if $\pi|_D : D \to S$ is surjective. If $\pi(D)$ is a point, we say that D is vertical.

Theorem A.9. *[Liu02, 9.1.12](Intersection on a fibered surface). Let $X \to S$ be a regular fibered surface. Let $s \in S$ be a closed point and denote by $Div_s(X)$ the set of divisors supported on the fiber X_s. Then there exists a unique bilinear map (of $\mathbb{Z}$-modules)*

$$i_s : Div\left(X\right) \times Div_s\left(X\right) \to \mathbb{Z}$$

which verifies the following properties:

(a) If $D \in Div\left(X\right)$ and $E \in Div_s\left(X\right)$ have no common component, then

$$i_s\left(D, E\right) = \sum_x i_x\left(D, E\right)\left[k\left(x\right) : k\left(s\right)\right],$$

where x runs through the closed points of X_s.

(b) The restriction of i_s to $Div_s\left(X\right) \times Div_s\left(X\right)$ is symmetric.

(c) $i_s\left(D, E\right) = i_s\left(D', E\right)$ if $D \sim D'$.

(d) If $0 \underset{\neq}{<} E \leq X_s$, then

$$i_s\left(D, E\right) = deg_{k(s)}\mathcal{O}_X\left(D\right)|_E.$$

Theorem A.10. *[Liu02, 9.1.37] Let $X \to S$ be a regular fibered surface, $s \in S$ a closed point, and $E \in Div_s\left(X\right)$ such that $0 \underset{\neq}{<} E \leq X_s$ (the second inequality is an empty condition if $dim\ S = 0$). Then we have*

$$\omega_{E/k(s)} \approx \left(\mathcal{O}_X\left(E\right) \otimes \omega_{X/S}\right)|_E,$$

and if $K_{X/S}$ is a canonical divisor,

$$p_a\left(E\right) = 1 + \frac{1}{2}\left(E^2 + K_{X/S} \cdot E\right).$$

Theorem A.11. *[Har77, III.7.7] Let X be a projective Cohen-Macaulay scheme of equidimension n over a field k. Then for any locally free sheaf $\mathscr{F}$ on X there are natural isomorphisms*

$$H^i\left(X, \mathscr{F}\right) \approx H^{n-i}\left(X, \mathscr{F}^\vee \otimes \omega_X^\circ\right)'$$

where ω_X° is the dualizing sheaf on X.

Finally note that the following Riemann-Roch formula holds over perfect fields. The proof is the same as the one in [Har77] Chapter 5, but the field is no longer assumed algebraically closed. All the necessary results are stated above.

Theorem A.12. *Let D be a divisor on a nonsingular, projective surface X over a perfect field k. Then*

$$\chi\left(\mathcal{O}_X\left(D\right)\right) = \frac{1}{2}D.\left(D-K\right)+\chi\left(\mathcal{O}_X\right).$$

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