Degrees of Generation

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Abstract

In [EgMMP], I show (among other things) the finite generation of the log canonical ring of an arithmetic threefold germ (i.e. a scheme of relative dimension 2 over a DVR. In this note, I examine the computability of degrees of generation for such rings, as well as exploring the effective basepoint-free theorem in certain cases. This was originally a section in [EgMMP] (c.f. version ~0.12) and has been separated since it was taking too long to review the other note (although the material is very similar).

1 Effective KLT Kawamata Viehweg Vanishing

In this section, I note that a modification of Tanaka's vanishing theorem [Tan2] gives a version of the (log) Kawamata Viehweg Vanishing in which the multiple of the nef divisor is at least computable. There is a similar recent result in [DF], which is more restrictive (although with a much better constant) in that it requires a smooth variety rather than a normal projective klt pair. First, a theorem of Terakawa:

Theorem 1. *[Ter] Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X. Assume that either*

(1) $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or

(2) X is of general type with

$p \geq 3$ and $(D^2) > vol(X)$ or

$p = 2$ and $(D^2) > max\{vol(X), vol(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then

$$H^i\left(X, \mathcal{O}_X\left(K_X + D\right)\right) = 0$$

for all $i > 0$.

Recall the following covering Lemma:

Lemma 2. *Let* X *be an* n*-dimensional smooth variety. Let* D *be a* $\mathbb{Q}$*-divisor such that the support of the fractional part* $\{D\}$ *is simple normal crossing. Moreover, suppose that, for the prime decomposition* $\{D\} = \sum_{i\in I} \frac{b^{(i)}}{a^{(i)}} D^{(i)}$, *no integers* $a^{(i)}$ *are divisible by* p. *Then there exists a finite surjective morphism* $\gamma : Y \to X$ *from a smooth variety* Y *with the following properties:*

(1) The field extension $K(Y)/K(X)$ *is a Galois extension.*

(2) $\gamma^* D$ *is a* $\mathbb{Z}$*-divisor (it seems the degree of the extension is an* m *clearing the denominators of the* $D^{(i)}$*.)*

(3) $\mathcal{O}_X\left(K_X + \ulcorner D \urcorner\right) \approx \left(\gamma_* \mathcal{O}_Y\left(K_Y + \gamma^* D\right)\right)^G$, *where* G *is the Galois group of* $K(Y)/K(X)$.

(4) If D' *is a* $\mathbb{Q}$*-divisor such that* $\{D'\} = \{D\}$, *then* $\gamma^* D'$ *is a* $\mathbb{Z}$*-divisor, and* $\mathcal{O}_X\left(K_X + \ulcorner D' \urcorner\right) \approx \left(\gamma_* \mathcal{O}_Y\left(K_Y + \gamma^* D'\right)\right)^G$.

Remark 3. Several aspects of the proof of the above Lemma will be used in the following.

Applying the above to Tanaka's proof seems to give the following:

Theorem 4. *([Tan2, 2.6] Weak Effective Kawamata-Viehweg Vanishing Theorem) Let* X *be a smooth projective surface over an algebraically closed field of characteristic* $p > 2$. *Let* N *be a nef and big* $\mathbb{R}$*-cartier and* B *a nef and big* $\mathbb{Q}$*-divisor whose fractional part is simple normal crossing, whose fractional part has no denominators divisible by the characteristic. Then there exists an* r, *computable in terms of the intersection numbers of the components of* B, N, *and* K_X, *such that*

$$H^i\left(X, \mathcal{O}_X\left(K_X + \ulcorner B \urcorner + rN + N'\right)\right) = 0$$

for every $i > 0$, *and every nef* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* N' *such that* $rN + N'$ *is a* $\mathbb{Z}$*-divisor.*

Proof. (A slightly different proof than the cited Theorem). The fractional part of $B + rN + N'$ is equal to the fraction part of B when $rN + N'$ is a

$\mathbb{Z}$-divisor. Thus we apply Lemma 2 to obtain a degree m cover $\gamma : Y \to X$ (independent of r and N') where Y is a smooth surface. Then

$$H^1\left(K_X + \ulcorner B \urcorner + rN + N'\right)$$

$$= H^1\left(K_X + \ulcorner B + rN + N' \urcorner\right)$$
$$= H^1\left(\gamma_*\left(K_Y + \gamma^*\left(B + rN + N'\right)\right)\right)^G$$
$$= H^1\left(Y, K_Y + \gamma^* B + r\gamma^* N + \gamma^* N'\right)^G.$$

the last term vanishing by Terakawa's Theorem and some $r \gg 0$, as by the proof of [KMM, 1-1-1], the cover can be made general type.

Now I claim it is possible to compute an r such that the last term above is zero using the intersections and degrees of components of $B = \sum a_i \Gamma_i$. It will suffice to show that Y is general type and that $(B + rN)^2 > K_Y^2$, so that Theorem 1 can be applied.

As B is big and nef, there exists an ample divisor A and an effective divisor D such that $B \equiv A - \frac{1}{j} D$ for all $j \gg 0$. Following the proof of [KMM, 1-1-1], let $M = kA$ be very ample (we can compute k by [DF]) and such that m clears the denominators of the components of B and such that $mM - \Gamma_i$ is very ample for all i (again we can compute such an m by [DF]: in the case that $\Gamma_i^2 < 0$, then ensure k is large enough so that $kA.\Gamma_i > -\Gamma_i^2$, take $\Gamma_i' = \Gamma_i + (k-1) K_X$ and by [DF, Theorem 1.2] find (computable) m' such that $m'M - \Gamma_i' = m''M - \Gamma_i$ is very ample. Replace m by m'', and letting $H_i \in |mM - \Gamma_i|$ for all i (thus $\frac{1}{m}(\Gamma_i + H_i) \sim M$ is very ample), we then have (again c.f. the proof of [KMM, 1-1-1])

$$K_Y$$

$$= \tau^* K_X + (m-1)\left(\sum \left(\tau^* \Gamma_i\right)_{red} + \sum \left(\tau^* H_k^{(i)}\right)_{red}\right)$$
$$= \tau^* K_X + \tau^*\left(\sum_i \left(\Gamma_i + H_i - \frac{1}{m}\left(\Gamma_i + H_i\right)\right)\right)$$
$$= \tau^* K_X + \tau^*\left(\left(m-1\right) M\right)$$

so that Y is general type, and $vol\,(Y) = \left[\left((m+1)\,k + 1\right) A\right]^2$, so it suffices to pick r large enough that $(B + rN)^2 > \left[\left((m+1)\,k+1\right) A\right]^2$ or, taking $j \gg 0$, such that $(B + rN) > \left[\left((m+1)\,k+1\right) B\right]^2$. □

Now I restate Tanaka's vanishing theorem, with a note on the computability in certain circumstances.

Theorem 5. *(Effective KLT Kawamata-Viehweg Vanishing for normal surfaces c.f. [Tan2, 2.11]) Let (X, Δ) be a normal projective klt surface over an algebraically closed field of characteristic $p > 2$, where $K_X + \Delta$ is an effective $\mathbb{R}$-divisor. Let N be a nef and big $\mathbb{R}$-cartier $\mathbb{R}$-divisor. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D - (K_X - \Delta)$ is nef and big. Then there exists an r_0* [1] *such that*

$$H^i\left(X, \mathcal{O}_X\left(D + rN + N'\right)\right) = 0$$

for every $i > 0$, every positive real number $r \geq r_0$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor N' such that $rN + N'$ is a Cartier divisor.

Proof. The computability follows easily from [Tan2, 2.11] except that Weak Kawamata Viehweg Vanishing in the proof is replaced by Theorem 4. Since, in this note I can apply this theorem when (X, Δ) is log smooth (hence X is smooth) and $\llcorner \Delta \lrcorner = 0$, here is the proof in that simple case:

$$H^1\left(D + rN + N'\right)$$

$$\begin{aligned} &= H^1\left(K_X + D - K_X - \llcorner \Delta \lrcorner + rN + N'\right) \\ &= H^1\left(K_X + D - K_X + \ulcorner -\Delta \urcorner + rN + N'\right) \\ &= H^1\left(K_X + \ulcorner D - K_X - \Delta \urcorner + rN + N'\right) \end{aligned}$$

since we can move the integral divisor $D - K_X$ into the round up. This last term vanishes by Theorem 4. The vanishing of H^2 follows easily using Serre Duality. □

Theorem 6. *[EgMMP] Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists a computable m_0 such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

[1] By Theorem 4 r is computable in terms of the number of components Δ and in terms of the self intersections of these components, and their multiplicities. Otherwise Tanaka's original statement gives just the existence of an r.

2 (Effective) Basepoint Freeness

This section is not necessary for the rest of the paper (and has been checked much less, although it the proof is essentially the same as in [EL], but substituting Theorem 4). The point is now that finite generation is achieved, can we get the specific degrees. It's possible this will be split off into another note in the future.

In [Fuj], Fujino proves the following Theorem:

Theorem 7. *Let (X, Δ) be a complete irreducible log surface and let D be a Cartier divisor on X. Put $A = D - (K_X + \Delta)$. Assume that A is nef, $A^2 > 4$, and $A.C \geq 2$ for every curve C on X such that $x \in C$. Then $\mathcal{O}_X(D)$ has a global section not vanishing at x.*

and conjectures that the same holds in characteristic $p > 0$.

In this section, I follow [EL] to prove a very weak version of the above conjecture:

Theorem 8. *Let (X, Δ) be a smooth projective klt surface over an algebraically closed field of characteristic $p > 2$, where Δ is an effective $\mathbb{R}$-divisor. Let D be a Cartier divisor on X. Put $A = D - (K_X + \Delta)$. Assume that A is nef. Then there exists computable m_0 and m_1 such that if $A^2 > m_0$, and $A.C \geq m_1$ for every curve C on X with $x \in C$. Then $\mathcal{O}_X(D)$ has a global section not vanishing at x.*

Proof. (Very Similar to [EL, 1.4]) By Riemann-Roch, and Theorem 5

$$h^0(n(B)) \approx n^2 B^2/2 \geq \frac{5n^2}{2}$$

so for $n \gg 0$, there exists for $n \gg 0$ $D \in |nB|$ with $q = mult_x(D) \geq 2n+1$ at x. Write $D = \sum d_i D_i + F$ with D_i the prime components of D meeting x.

If $q > 2d_i$ for all i, let $f : Y \to S$ the blowing-up of x, let $E \subset Y$ the exceptional divisor, and let $D_i' \subset Y$ the proper transform of D_i. Using standard formula for blow-ups on surfaces

$$f^* D = \sum d_i D_i' + qE + f^* F.$$

Let

$$M = f^* B - \frac{2}{q} f^* D$$

$$= f^*B - \frac{2}{q}\left[\sum d_i D_i' + qE + f^*F\right]$$

$$= f^*B - 2E - \sum \frac{2d_i}{q} D_i' - \frac{2}{q} f^*F$$

on Y. As $\frac{2d_i}{q} < 1$, the D_i' term rounds up to zero, thus

$$\ulcorner M \urcorner = f^*B - 2E - \underbrace{\ulcorner \frac{2}{q} f^*F \urcorner}_{N}$$

where N is effective and doesn't meet E. Thus

$$K_Y + \ulcorner M \urcorner$$

$$= \underbrace{f^*(K_S) + E}_{K_Y} + f^*B - 2E - N$$

$$= f^*(K_S + B) - E - N.$$

As $M = f^*B - \frac{2}{q} f^*D \equiv \left(1 - \frac{2n}{q}\right) f^*B$ and $q > 2n$, so $1 - \frac{2n}{q} > 0$, then M is nef and big. Now to apply Theorem 4, find some computable m_0, m_1 such that the self intersection of M is large enough. Then

$$H^1(\mathcal{O}_S(K_S + B - f_*N) \otimes \mathfrak{m}_x) = H^1(K_Y + \ulcorner M \urcorner) = 0.$$

On the other hand, if $q < 2d_i$ for some i, but $q = \sum d_i \cdot mult_x(D_i)$ then $q \geq d_i$ for all i. As in the cited proof, let D_0 the unique component with maximal multiplicity $d_0 > \frac{q}{2}$, which must be smooth at x. Let $M = B - D/d_0$. Then $M \equiv (1 - n/d_0) B$ is nef and big as above and

$$K_S + \ulcorner M \urcorner = K_S + B - D_0 - N$$

where again N is supported away from x. Again applying Theorem 4 (enforcing B to have large enough computable intersection numbers),

$$H^0(K_S + B - N) \to H^0((K_S + B - N)|_{D_0})$$

is surjective. The rest follows as in the cited Theorem. (possibly filled out next update). $\square$

3 Finite Generation

Corollary 9. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. After possibly extending R, the rings $R(K_X + \Delta)$ and $R(K_X + \Delta, K_X + \Delta + A)$ are generated over R in degree m_0, for any big A with $K_X + \Delta + A$ nef, where m_0 is a computable constant.*

Proof. By Theorem 8, since (X_k, Δ_k) has a minimal model by [Tan2], there is computable $m \gg 0$, such that

$$\bigoplus_{m \geq 0} H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right)$$

is generated in degree 1, so that there is a surjection

$$S^l H^0\left(m\left(K_{X_k} + \Delta_k\right)\right) \twoheadrightarrow H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right).$$

Consider the following diagram:

$$\begin{array}{ccc} S^l H^0\left(m\left(K_X + \Delta\right)/R\right) & \xrightarrow{\alpha} & H^0\left(ml\left(K_X + \Delta\right)/R\right) \\ \downarrow & & \downarrow \\ S^l H^0\left(m\left(K_{X_k} + \Delta_k\right)\right) & \xrightarrow{\twoheadrightarrow} & H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right) \end{array}$$

The vertical maps are surjective by the first part of the theorem (and applying Nakayama's lemma and cohomology and base change to pull back the generators of the k-modules). Now applying the version of Nakayama's Lemma given in [GH, Chapter 5.3], surjectivity of the bottom map implies surjectivity of α, and thus $R(K_X + \Delta)$ is finitely generated over R.

Similar logic applies to $R(K_X + \Delta, K_X + \Delta + A)$, where the minimal model for $K_{X_k} + \Delta_k$ is achieved by scaling of A_k in this case (the minimal model program with scaling for surfaces in positive characteristic is given in [EgMMP] □

References

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