LOG MINIMAL MODELS FOR ARITHMETIC THREEFOLD GERMS

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Abstract. In this note I study the existence of log minimal models for a pair of relative dimension 2 over a discrete valuation ring possibly having mixed characteristic. This generalizes the semistable result of Kawamata [Kaw1994, Kaw1998]. Also I note the invariance of log plurigenera for such pairs, generalizing the result of [Suh].

Contents

1. Introduction

In this paper I prove the following Theorem

Theorem 1. *Let* (X, Δ) *be a klt pair of relative dimension* 2*, proper over a DVR* R *with perfect residue field* k *of characteristic* $p > 0$ *and perfect fraction field* K*. Assume that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier, and log smooth over* R*. Then the minimal model of* (X, Δ) *exists.*

This is somewhat related to a theorem of Kawamata [Kaw1994, Kaw1998] where he proves the semistable case of the above assuming no boundary and in characteristics $p \geq 5$. It is also somewhat related to the recent proofs of the existence of minimal models in positive characteristic for threefolds in [HX, Bir], (here we have geometric dimension 2 and arithmetic dimension 1, rather than geometric dimension 3 and arithmetic dimension 0). The above Theorem is the result of applying the techniques of [CL] (in the big case) and [HMX2014] (in general) after achieving a generalization of Suh's Theorem on the invariance of plurigenera [Suh] (namely to Kawamata Log Terminal Pairs). Thanks to my advisor, Professor Hacon for help in catching bugs and helpful suggestions.

2. Intersection Theory on X/R

Suppose X/R is a proper algebraic space of relative dimension 2 over a discrete valuation ring R with residue field k of characteristic $p > 0$ and perfect fraction field K (See for example the compact Shimura Varieties mentioned in [Suh, 0.1]).

Lemma 2. [KU, 9.3] *Let $\varphi : X \to Spec(R)$ be an algebraic two-dimensional space, smooth, proper, separated, and of finite type over $spec(R)$, where R is a DVR with algebraically closed residue field k, and field of fractions K. Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ. If D, D' are divisors on X, then*

$$K^2_{X_k} = K^2_{X_K}$$
$$(K_{X_k} \cdot D_k)_{X_k} = (K_{X_K} \cdot D_K)_{X_K}$$
$$(D_k \cdot D'_k)_{X_k} = (D_K \cdot D'_K)_{X_K}.$$

If X/R is smooth, then Lemma 2 applies to show that for any divisors C, D extending to both fibers, we can define $C.D = (C|_{X_k}.D|_{X_k})_{X_k}$. On the other hand, if X/R is merely normal and proper, but is actually a scheme, the resolution of singularities, Theorem 4, holds. Thus the intersection theory can be defined as in [Tan, Def 3.1]: $f : X' \to X$ is a resolution, and $C.D = f^*C.f^*D$ for two divisors C, D extending to both fibers of Y/R. By properness, this intersection extends by linearity to Weil divisors with $\mathbb{Q}$ or $\mathbb{R}$ coefficients. Numerical equivalence and $N^1(X)_{\mathbb{Q},\mathbb{R}}$ are then defined as usual.

Note the two different hypothesis here: The invariance of plurigenera theorem requires a proper, smooth algebraic space of relative dimension

2, while the existence of minimal models requires the intermediate steps of the minimal model program to have some intersection theory, and thus be schemes. Recall that a Cartier divisor C on a variety is usually called nef if $C.D \geq 0$ for all curves C.

Lemma 3. *Let S be an affine Dedekind scheme and $f : X \to S$ a projective morphism. Let $\mathcal{L}$ be an invertible sheaf on X such that $\mathcal{L}_s$ is nef for every closed point $s \in S$. Then $\mathcal{L}$ is nef.*

Proof. By [Liu, 5.3.24], the statement holds when nef is replaced by ample. Now, c.f. [Laz, 1.4.10], it suffices to choose an ample divisor H which restricts to X_s, so that $D_s + \epsilon H_s$ is ample for all sufficiently small ϵ. Then $D + \epsilon H$ is ample for all sufficiently small ϵ, and so D is nef. □

3. Log Resolution

The following statement of resolution will be used. So for example, if X is an algebraic variety over a number field K then, by [MP, III.2], there is a completed arithmetical variety $\overline{X}$ of dimension $dim\,(X)+1$:

$$\overline{X} \to \overline{Spec\ \mathcal{O}_K}.$$

(Note that a complete DVR is excellent).

Theorem 4. [CP2014, 1.1] *Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*

(i) X' *is everywhere regular*

(ii) π *induces an isomorphism* $\pi^{-1}(Reg\ X) \approx Reg\ X$

(iii) $\pi^{-1}(Sing\ X)$ *is a strict normal crossings divisor on* X'.

The following Statement of Log Resolution will be used in the Log Smooth case (i.e. the case with (X, Δ) a pair with X smooth and Δ simple normal crossings.)

Theorem 5. [CP2014, 4.3] *Let* S *be a regular Noetherian irreducible scheme of dimension three which is excellent and* $\mathcal{I} \subset \mathcal{O}_S$ *be a nonzero ideal sheaf. There exists a finite sequence*

$$S =: S(0) \leftarrow S(1) \leftarrow \cdots \leftarrow S(r)$$

with the following properties:

(i) for each j, $0 \leq j \leq r-1$, $S(j+1)$ *is the blowing up along a regular integral subscheme* $\mathcal{Y}(j) \subset \mathcal{S}(j)$ *with*

$$\mathcal{Y}(j) \subseteq \left\{ s_j \in \mathcal{S}(j) : \mathcal{I}\mathcal{O}_{S(j),s_j} \text{ is not locally principal} \right\}.$$

(ii) $\mathcal{I}\mathcal{O}_{S(r)}$ *is locally principal.*

4. KLT Kawamata Viehweg Vanishing

Theorem 6. [Tan2, 2.11] *Let* (X, Δ) *be a normal projective klt surface over an algebraically closed field of characteristic* $p > 0$, *where* $K_X + \Delta$ *is an effective* $\mathbb{R}$*-divisor. Let* N *be a nef and big* $\mathbb{R}$*-cartier* $\mathbb{R}$*-divisor. Let* D *be a* $\mathbb{Q}$*-Cartier* $\mathbb{Z}$*-divisor such that* $D - (K_X - \Delta)$ *is nef and big.*

Then there exists an r_0 such that

$$H^i\left(X, \mathcal{O}_X\left(D + rN + N'\right)\right) = 0$$

for every $i > 0$, every positive real number $r \geq r_0$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor N' such that $rN + N'$ is a Cartier divisor.

Remark 7. Using Terakawa's Theorem [Ter], (and possibly restricting the characteristic to be greater than 2) it is possible to prove a version of Theorem 6 where the constant r_0 is computable. A sketch appears on my blog.

5. Invariance of Plurigenera Special Case

Lemma 8. [KU, 9.4] *Let $\varphi : X \to Spec\,(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $spec\,(R)$, where R is a DVR with algebraically closed residue field k, and field of fractions K. Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ. If X_k contains an exceptional curve of the first kind e, there exists a DVR $\tilde{R} \supset R$, with residue field isomorphic to k, and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \to \tilde{Y}$ over $Spec\left(\tilde{R}\right)$ such that on the closed fibre, π induces the contraction of the exceptional curve e. Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.*

Proof. (Follows easily from the cited theorem, although I make a small observation about the extension of residue fields). By [A1, Cor 6.2]

$Hilb_{X/Spec(R)}$ is represented by an algebraic space $\mathscr{H}$ which is locally of finite type over $Spec\,(R)$. Let Y be the irreducible component containing the point $\{e\}$ corresponding to the exceptional curve e on the special fiber. Then $e \approx \mathbb{P}^1_k$ and $N_{e/X} \approx \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, so Y is regular at $\{e\}$ and of dimension 1. Since e is fixed in the special fiber, the structure morphism $p : Y \to Spec\,(R)$ is surjective. By [Bo, 2.2.14][1], we can find an étale cover $\tilde{R} \supset R$ and a morphism $j : Spec\left(\tilde{R}\right) \to Y$ over $Spec\,(R)$ with $j\,(\tilde{o}) = \{e\}$ (if R is not already complete, then first extend R to a complete DVR using [Gr, 0.6.8.2,3] so that $\tilde{R}$ is again a DVR). As k is assumed algebraically closed, and $\tilde{R} \to R$ is unramified, then the extension of residue fields is finite and separable at the closed point of R, and hence an isomorphism of residue fields. Let $\hat{p} : \mathscr{E} \to Spec\left(\tilde{R}\right)$ be the pull-back of the universal family over Y. As the closed fibre $\mathscr{E}_0$ is a projective line, we may choose the morphism j in such a way that the generic fibre $\mathscr{E}_1$ of $\hat{p}$ is also a projective line. Moreover, $\mathscr{E}$ can be considered as a smooth closed algebraic subspace of codimension 1 in $\hat{X} = X \otimes \tilde{R}$. By Lemma 2, $-1 = \mathscr{E}_0^2 = \mathscr{E}_1^2$. Hence $\mathscr{E}_1$ is also an exceptional curve of the first kind. Hence by [A2, Cor 6.11] there exists a contraction morphism $\pi : \hat{X} \to \hat{X}$ over $Spec\left(\tilde{R}\right)$ which contracts $\mathscr{E}$ to a section of $\tilde{\varphi} : \tilde{X} \to Spec\,(R)$ where $\tilde{\varphi}$ is proper, smooth, separated and of finite type over $Spec\left(\tilde{R}\right)$. □

Lemma 9. [KK94, 2.3.5] *Let (S, B) be a log-canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a*

[1] If k is perfect, the proof of the cited lemma goes seems to go through when $f : X \to S$ is merely smooth at $x \in X$ and $f : X \to S$ is locally of finite type.

curve with $C^2 < 0$ *and* $C.(K_S + B) < 0$*, then* $C \approx \mathbb{P}^1$ *and it can be contracted to a log-canonical point.*

Theorem 10. [Tan, 4.4, 4.6] *Let* X *be a projective normal surface and let* Δ *be an effective* $\mathbb{R}$*-divisor such that* $K_X + \Delta$ *is* $\mathbb{R}$*-cartier. Let* $\Delta = \sum b_i B_i$ *be the prime decomposition. Let* H *be an* $\mathbb{R}$*-cartier ample* $\mathbb{R}$*-divisor. Then the following assertions hold:*

(1) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$

(2) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i]$

(3) Each C_i *in (1) and (2) is rational or* $C_i = B_j$ *for some* B_j *with* $B_j^2 < 0$.

(4) Each C_i *in (1) and (2) satisfies* $0 < -C_i.(K_X + \Delta) \leq 3$.

Theorem 11. [KK94, 2.3.6 (Log MMP For Surfaces)] *Let* (S, B) *be a log canonical surface (over an algebraically closed field). There exists a sequence of contractions* $f : S \to S_1 \to \cdots \to S_n = S'$ *such that* S' *is log canonical (even plt at every point where* f^{-1} *is not an isomorphism) and satisfies exactly one of the following conditions:*

- $K_{S'} + f_*B$ *is nef*
- *There exists a* $g : S' \to T$ *morphism, such that* S' *is a birationally ruled surface over the curve* T.
- (S', f_*B) *is a log Del Pezzo surface.*

Remark 12. (Facts about terminal pairs) If $dim\ X = 2$ and (X, D) is a terminal pair, then X is smooth[Ko2013]. By [KM, 3.43], terminal singularities are preserved after birational contraction of a $K_X + D$ negative extremal ray, as long as the ray is not a component of D.

Proposition 13. *Let (X,Δ) be a terminal pair of relative dimension 2 over a DVR R with algebraically closed residue field k of characteristic $p>0$ and perfect fraction field K. Assume that $\mathbb{B}_{-}(K_X+\Delta)\wedge\Delta=\emptyset$ and that $K_X+\Delta$ is $\mathbb{Q}$-Cartier with $\kappa(K_{X_k}+\Delta_k)=2$. Then there exists an m_0 such that for $m>m_0$,*

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Delta_K\right)\right).$$

The proof of proposition 13 is given in the following claims.

Claim. Assumptions as above, after passing to an extension R' of R, there is a proper, smooth algebraic space X^{min}/R' and an R' morphism $X\otimes R'\to X^{min}$ such that both $X_k\to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}}+\Delta_{X_k^{min}}$ is nef.

Proof. As k is algebraically closed of characteristic $p>0$, then by the Cone Theorem 10,

$$\overline{NE}\left(X_k\right)=\overline{NE}\left(X_k\right)_{K_{X_k}+\Delta_k\geq 0}+\sum\mathbb{R}_{\geq 0}\left[C_i\right]$$

with each C_i is rational or $C_i=B_j$ for some B_j a component of Δ with $B_j^2<0$. Under the assumption that $\kappa(K_{X_k}+\Delta_k)=2$, by Theorem 11 then it must be the case that $C_i^2<0$. Thus Theorem 9 implies $C_i\approx\mathbb{P}^1$ and C_i can be contracted to a log-canonical point. By Remark 12 (which applies since $\mathbb{B}_{-}(K_X+\Delta)\wedge\Delta=\emptyset$) the resulting pair is again terminal, hence smooth, and so it must be the case that C_i was an exceptional curve of the first kind. Therefore it is possible to apply Lemma 8.

By Lemma 8, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \to \tilde{X}$ induces the contraction of C_i on X_k and X_K, and further $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since k is algebraically closed, it induces identity on residue fields. Now I need to work with $\tilde{X}$. $\tilde{X}_k$ is projective, so the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no problem in extending R. As the Picard number of the special fiber drops each step, there are only finitely many steps. □

Claim 14. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef.

Proof. It suffices to apply Lemma 3 to X_k^{min}, and then restrict to X_K. □

Claim 15. There is an m_0 such that for $m_0|m$, we have

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Proof. Suppose first that (X_k, Δ_k) is big. By the claims 5 and 14, we achieve X^{min} such that $K_X + \Delta$ is nef on both X_k^{min} and X_K^{min}. By Theorem 6 applied to the special fiber (and the semicontinuity theorem) there exists an $m_0 \gg 0$ such that for $m_0 < m$ and $i > 0$, we have

$$h^i\left(m\left(K_{X_K} + \Delta_K\right)\right) = h^i\left(m\left(K_{X_k} + \Delta_k\right)\right) = 0.$$

Thus, applying the invariance of Euler characteristic [Liu, 5.3.22], and birational invariance of the plurigenera, it follows that for $m > m_0$,

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Delta_K\right)\right).$$

□

Remark 16. In fact, if $\Delta = 0$, then, at the end of the above proof, Ekedahl's vanishing Theorem [Ek] can be applied and $m_0 = 2$, in general m_0 can be computed by Theorem 6.

6. Invariance of Plurigenera General Case

Theorem 17. *Let (X, Δ) be a log smooth klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is big, and $\mathbb{Q}$-Cartier. Then, there exists an m_0 such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Delta_K\right)\right).$$

Proof. Since the hypothesis and conclusion are preserved by base change, then after extending R if necessary, we may assume that k is algebraically closed and that R is complete. To begin, I make the reduction of [HMX2010, 1.6]. As (X, Δ) is log smooth, then replacing (X, Δ) by a blow-up of strata, assume (X, Δ) is terminal. Recall that the log-canonical ring of $K_{X_k} + \Delta_k$ is finitely generated (c.f. [Tan, 7.1]) so $N_\sigma\left(K_{X_k} + \Delta_k\right)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma\left(K_{X_k}+\Delta_k\right)$$

and let $0 \leq \Theta \leq \Delta$ be a $\mathbb{Q}$-divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. Letting $m \gg 0$ to fit the hypothesis of Proposition 13, and sufficiently divisible such that $m\left(K_{X_k} + \Theta_k\right)$ is integral, then by definition of N_σ,

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_k} + \Theta_k\right)\right).$$

Furthermore, by Proposition 13,

$$h^0\left(m\left(K_{X_k} + \Theta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Theta_K\right)\right).$$

As $\Theta \leq \Delta$, the theorem follows by semicontinuity. □

7. Finite Generation (Big Case)

Corollary 18. *Situation as above, after possibly extending R, the canonical ring $R\left(K_X + \Delta\right)$ is finitely generated over R.*

Proof. By [Tan], there is $m \gg 0$, such that

$$\bigoplus_{m \geq 0} H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right)$$

is generated in degree 1, so that there is a surjection

$$S^l H^0\left(m\left(K_{X_k} + \Delta_k\right)\right) \twoheadrightarrow H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right).$$

Consider the following diagram:

$$\begin{array}{ccc} S^l H^0\left(m\left(K_X + \Delta\right)/R\right) & \xrightarrow{\alpha} & H^0\left(ml\left(K_X + \Delta\right)/R\right) \\ \downarrow & & \downarrow \\ S^l H^0\left(m\left(K_{X_k} + \Delta_k\right)\right) & \xrightarrow{\twoheadrightarrow} & H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right) \end{array}$$

The vertical maps are surjective by the first part of the theorem (and applying Nakayama's lemma and cohomology and base change to pull back the generators of the k-modules). Now applying the version of Nakayama's Lemma given in [GH, Chapter 5.3], surjectivity of the bottom map implies surjectivity of α, and thus $R(K_X+\Delta)$ is finitely generated over R. $\square$

Definition 19. Let $(X, \sum S_i)$ a log smooth, log-canonical projective pair where X is a two dimensional, normal variety over a perfect field of characteristic $p>0$. with S_i distinct prime divisors, $V=\sum \mathbb{R}S_i \subset Div_{\mathbb{R}}(X)$. Define the following sets, the first of which is clearly a polytope.

$$\mathcal{L}(V)=\left\{\sum_{i=1}^{n} a_i S_i \in V \mid a_i \in [0,1]\right\}$$
$$\mathcal{E}(V)=\left\{\Delta \in \mathcal{L}(V) \mid K_X+\Delta \sim_{\mathbb{R}} D \geq 0\right\}.$$

Given $f: X \to Y$ a birational contraction, let $\mathcal{C}_f(V)$ denote the closure of $\mathcal{L}(V)$ of

$$\left\{\Delta \in \mathcal{E}(V) \mid f \text{ is a log terminal model of } (X,\Delta)\right\}.$$

C.f. [CL2013, 2.13], (the statement is for characteristic 0 in any dimension by [SC, 3.4], but in the surface case here merely relies on [Tan, 0.2]) then there are birational contractions $f_i : X \to Y_i$ such that $\mathcal{C}_{f_i}(V), ..., \mathcal{C}_{f_k}(V)$ are rational polytopes and

$$\mathcal{E}(V)=\bigcup_{i=1}^{k} \mathcal{C}_{f_i}(V)$$

so that $\mathcal{E}(V)$ is also a polytope.

Proposition 20. [H] *Let X be a two dimensional $\mathbb{Q}$-factorial normal variety over a perfect field of characteristic $p>0$. Let $\{(X,\Delta_i)\}_{i\in\{1,...,k\}}$ be a big, log smooth, $\mathbb{Q}$-Cartier KLT pair for each i and such that $K_X+\sum\Delta_i$ has simple normal crossings support. Then the ring*

$$R(X,K_X+\Delta_1,...,K_X+\Delta_k)$$

is finitely generated.

Corollary 21. *Let $\{(X,\Delta_i)\}_{i\in\{1,...,n\}}$ be a collection of klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p>0$ and perfect fraction field K. Assume that $K_X+\Delta_i$ is big, $\mathbb{Q}$-Cartier and such that $K_X+\sum\Delta_i$ has simple normal crossings support. Then the ring*

$$R(X,K_X+\Delta_1,...,K_X+\Delta_n)$$

is finitely generated over R.

Proof. C.f. the proof of [CZ, 2.11], it suffices to prove that if $G=\sum_{i=1}^{k}m_i(a(K_X+\Delta_i))$, with fixed $a\in\mathbb{N}$, $m_1,...,m_k\geq0$, and $l\in\{1,...,l\}$ then there exists m_l such that the following maps are surjective

$$H^0(X,\mathcal{O}_X(G-a(K_X+\Delta_i)))\otimes H^0(X,\mathcal{O}_X(a(K_X+\Delta_i)))$$

$$\to H^0(X,\mathcal{O}_X(G)).$$

By applying Nakayama's Lemma and Theorem 17 and Proposition **??** as in the proof of Theorem 18, it suffices to show the surjectivity on the central fiber which holds by proposition 20. □

8. Finiteness of Models For Surfaces

The following theorem is not anything new, but is stated here as the minimal model program is not usually run with scaling on a surface (in [KK94, Tan], termination is proven without scaling).

Theorem 22. *Let X be a two dimensional normal variety over a perfect field of characteristic $p > 0$. Let (X, Δ) be a log smooth KLT pair and A an ample $\mathbb{Q}$-divisor on X with a good minimal model (which exists for non-negative Kodaira dimensions by [Tan]). Then there exists an $\epsilon > 0$ such that the minimal models (and the output of the minimal model program with scaling) for $(X, \Delta + tA)$ are all isomorphic for $t \in [0, \epsilon]$.*

Proof. (Essentially same as [Lai, 26,27] and simplifies due to the fact that a birational isomorphism in codimension 1 on a surface gives an isomorphism). Let $\phi : (X, \Delta) \to (X_g, \Delta_g)$ be a good minimal model. Let $f : X_g \to Z = Proj\ R\left(K_{X_g} + \Delta_g\right)$ the contraction (which exists by since $K_{X_g} + \Delta_g$ is nef, hence semiample by [Tan]). Then ϕ contracts the divisorial part of $\mathbb{B}\left(K_X + \Delta\right)$. Pick $t_0 > 0$ such that $(X_g, \Delta_g + t_0\phi_*A) = (X_g, \Delta_g + t_0A_g)$ is klt with A ample on X (note A_g is big, not in general nef). For H ample on X_g, let $\phi : X_g \to X'$ be the minimal model program with scaling, which terminates (again by [Tan]), and gives a minimal model of $(X_g, \Delta_g + t_0\Delta_g)$

over Z. For any curve contracted by f, $\left(K_{X_g}+\Delta_g\right).C=0$, hence $K_{X'}+\psi_*\Delta_g = K_{X'}+\Delta' \equiv_Z 0$. Thus curves contracted by ψ have trivial intersection with $K_{X_g}+\Delta_g$, and intersect negatively with A_g. Thus changing t does not affect which curves intersect negatively with $K_{X_g}+\Delta_g+tA_g$, and so X' is a minimal model of (X_g,Δ_g+tA_g) for all $t\in(0,t_0]$.

Now $\Delta'+t_0A'$ with $A'=\psi_*A_g$ (which is big) implies by the Theorem 10, that there exist only finitely many $K_{X'}+\Delta'+t_0A'$ negative extremal rays in $\overline{NE}\left(X'\right)$. These are all necessarily just intersecting A' negatively, so decreasing t_0, eventually we get to a point where a further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (Now we are on X', not over Z). Again shrinking t_0, suppose that $\psi\circ\phi$ is discrepancy negative w'r't $(X,\Delta+tA)$ for all t in the half open interval $(0,t_0]$. Note that $\mathbb{B}\left(K_X+\Delta+t_0A\right)\subset\mathbb{B}\left(K_X+\Delta\right)$, so ψ contracts only the curves not contracted by ϕ. Thus $\psi\circ\phi$ is discrepancy negative on the closed interval $[0,t_0]$, so X' is a minimal model of $(X,\Delta+tA)$ for all $t\in[0,t_0]$. Thus $\mathbb{B}\left(K_X+\Delta+tA\right)$ has the same divisorial components for all $t\in[0,t_0]$. Hence, any two minimal models for different $t\in[0,t_0]$ are birational and isomorphic in codimension one, and as the total dimension is two, they are thus isomorphic. $\square$

9. Big Abundance / R

Definition 23. Let $D_1,...,D_r$ $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. D_i are called **adjoint divisors** on X if they are of the form $D_i=K_X+\Delta_i$ for some

pair (X, Δ_i) where X is normal and projective, $\Delta_i \geq 0$ and is a $\mathbb{Q}$-divisor. Let

$$R = R(X; D_1, ..., D_r) = \bigoplus_{(n_1,...,n_r)\in\mathbb{N}^r} H^0(X, n_1 D_1 + \cdots + n_r D_r).$$

and define the **support** of R to be

$$Supp\ R = \left(\sum_{i=1}^{r} \mathbb{R}_+ D_i\right) \cap Div_{\mathbb{R}}^{eff}(X).$$

Theorem 24. [CL, 3.5] *Let X be a normal scheme of relative dimension 2, proper over S, the spectrum of a DVR which has perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier divisors on X. Assume that $R(X; D_1, ..., D_r)$ is finitely generated. Let $D : \mathbb{R}^r \ni (\lambda_1, ..., \lambda_r) \mapsto \sum \lambda_i D_i \in Div_{\mathbb{R}}(X)$ be the tautological map.*

(1) The support of R is a rational polyhedral cone.

(2) If $Supp\ R$ contains a big divisor and $D \in \sum \mathbb{R}_+ D_i$ is pseudoeffective, then $D \in Supp\ R$.

(3) There is a finite rational polyhedral subdivision $Supp\ R = \bigcup C_i$ such that σ_Γ is a linear function on C_i for every geometric valuation Γ of X. Furthermore, there is a coarsest subdivision with this property, in the sense that, if i and j are distinct, there is at least one geometric valuation Γ of X such that (the linear extensions of) $\sigma_\Gamma|_{C_i}$ and $\sigma_\Gamma|_{C_j}$ are different.

(4) There is a finite index subgroup $\mathbb{L} \subset \mathbb{Z}^r$ such that for all $n \in \mathbb{N}^r \cap \mathbb{L}$, if $D(n) \in Supp\ R$, then $\sigma_\Gamma(D(n)) = mult_\Gamma |D(n)|$.

Proof. (1, 2) follow easily from the cited theorem. For $3, 4$ note that [ELM, 4.7] is stated for an arbitrary Noetherian scheme, and this theorem is the only result used in the proof of this result which is given in [CL, 3.5]. □

Theorem 25. *([CL, 3.6]) Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier that $R(K_X + \Delta, K_X + \Delta + A)$ is finitely generated. If $K_X + \Delta$ is pseudo-effective, then it is $\mathbb{Q}$-effective.*

Proof. As in the source paper.

□

Let Γ a geometric valuation on X. Define

$$\sigma_\Gamma = inf\,\{mult_\Gamma D' \mid D \sim_{\mathbb{R}} D' \geq 0\} \in \mathbb{R}.$$

As a result of Theorem 21, there is the following abundance theorem:

Corollary 26. [CL, 3.8] *Let (X, Δ) be a big klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $R(K_X + \Delta)$ and $R(K_X + \Delta, K_X + \Delta + A)$ are finitely generated for an ample $\mathbb{Q}$-divisor A on X. If $K_X + \Delta$ is nef, then it is semi-ample.*

Proof. (This simplifies from the cited theorem as I assume the pair $K_X + \Delta$ is big). For any $\epsilon > 0$, and any geometric valuation Γ on X, as $K_X + \Delta + \epsilon A$ is ample a positive multiple is basepoint free and thus,

$$\sigma_\Gamma (K_X + \Delta + \epsilon A) = 0$$

so that σ_Γ is identically zero, and thus the centre of Γ is not in $\mathbb{B}(K_X + \Delta)$ for any such Γ. $\square$

Corollary 27. *Let X be a normal projective variety of dimension 2 (over a perfect field k) and let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Assume that the ring $R = R(X; D_1, ..., D_r)$ is finitely generated, and let Supp $R = \bigcup C_i$ be a finite rational polyhedral subdivision such that for every geometric valuation Γ of X, σ_Γ is linear on C_i, as in Theorem 24. Denote $\varphi : \sum \mathbb{R}_+ D_i \to N^1(X)_\mathbb{R}$ the projection, and assume there exists k such that $C_k \cap \varphi^{-1}(Amp\ D) \neq \emptyset$. Then $C_k \subset Supp\ R \cap \varphi^{-1}(Nef\ X)$. If the subdivision is coarsest, then $\mathcal{C}_k = Supp\ R \cap \varphi^{-1}(Nef\ X)$*

Proof. As in the source, but applying Theorems 24 and 26. $\square$

10. Cone, Rationality, and contraction / R

Definition 28. [CL, def 4.1] Let W a finite dimensional real vector space, $C \subset W$ a closed convex cone spanning W, and $v \in W$. The visible boundary of C from v is

$$V = \{w \in \partial C \mid [v, w] \cap C = \{w\}\}.$$

Corollary 29. *(c.f. [CL, 4.2]) (Kawamata's Rationality, Cone and Contraction Theorem) Let (X, Δ) be a pair of relative dimension 2, proper over a DVR R with perfect residue field k of characteristic*

$p>0$ and perfect fraction field K. Assume that $v_0=K_X+\Delta$ is $\mathbb{Q}$-Cartier and is obtained from the log smooth klt pair (X',Δ') after a finite number of steps of the minimal model program with scaling associated to a big divisor A with $K_X+\Delta+tA$ nef on X/R for some $t>0$. Let V be the visible boundary of $Nef\,(X/R)$ from $v_0\in N^1(X)_{\mathbb{R}}$ and $u_0\in V\cap(v_0,[K_X+\Delta+tA])$. Then there exists a locally polyhedral neighborhood $U\subset V$ of u_0 such that $\forall u'\in U$, u' is semiample.

Proof. This is essentially a local version of the cited theorem. Let $\mathcal{C}=\mathbb{R}_+v_0+Nef\,(X/R)$. Let u_0' be a rational point of $N^1(X/R)$ sufficiently close to u_0 such that $u_0\in B(u_0',\epsilon)$ is a sup-norm ball with rational vertices $w_j=K_X+\Delta+A_j,\ j=1,...,r$ which are klt and pull back to big log smooth klt pairs on X', and such that ϵ is small enough that any $K_X+\Delta+A'\in B(u_0',\epsilon)$ pulls back to a log smooth divisor on X'. Let $\mathcal{B}$ denote the convex hull of $B(u_0',\epsilon)$. Then by Theorem 21, the ring

$$R=R\left(X,K_X+\Delta+A_1,...,K_X+\Delta+A_r\right)$$

is finitely generated. Let $\varphi:\sum\mathbb{R}_+(K_X+\Delta+A_j)\to N^1(X)_{\mathbb{R}}$ the natural projection so that $\varphi\left(\sum\mathbb{R}_+(K_X+\Delta+A_j)\right)=\mathbb{R}_+\mathcal{B}$. Theorem 24 implies that *Supp* R is a rational polyhedral cone which intersects the interior of the nef cone. Therefore, by Theorem 24(2), if U is the portion of $\partial Nef\,(X/R)$ contained in $B(u_0',\epsilon)$, then $\varphi^{-1}(U)\subset$ *Supp* (R). Let *Supp* $R=\bigcup\mathcal{L}_k$ be the coarsest subdivision given by Theorem 24(3). For some k, $\mathcal{L}_k\cap\varphi^{-1}(Amp\ X)\neq\emptyset$, so by Theorem 27, $\mathcal{L}_k=Supp\ R\cap\varphi^{-1}(Nef\,(X))$, so that U is locally polyhedral. If $u'\in U$, with divisor D then there exists an ample $\mathbb{Q}$-divisor A' such that

$D \sim_{\mathbb{Q}} K_X + \Delta + A'$. By choice of ϵ, D pulls back to a log smooth divisor on X', and so that $R\left(K_X + \Delta + A', K_X + \Delta + A' + A''\right)$ is finitely generated for any ample A'' such that $D + A'' \in B\left(u_0', \epsilon\right)$. Thus Theorem 26 gives that u' is semiample. □

Lemma 30. [CL, Lemma 5.1]*Let X and Y be $\mathbb{Q}$-factorial projective schemes smooth over a DVR R with algebraically closed residue field. Let $f : X \to Y$ be a birational contraction, and let $\tilde{f} : \mathbb{K}(Y) \approx \mathbb{K}(X)$ the induced isomorphism on function fields. Then:*

(1) $f_* div_X \varphi = div_Y f(\varphi)$ for every $\varphi \in \mathbb{K}(X)$;

(2) for every geometric valuation Γ on $\mathbb{K}(X)$, and for every $\varphi \in \mathbb{K}(X)$, we have $mult_\Gamma(div_X \varphi) = mult_\Gamma\left(div_Y \tilde{f}(\varphi)\right)$;

(3)if f is an isomorphism in codimension one, then $f_* : Div_{\mathbb{R}}(X) \to Div_{\mathbb{R}}(Y)$ is an isomorphism, and for every $D \in Div_{\mathbb{R}}(X)$, there is an isomorphism $H^0(X, D) \approx H^0(Y, f_* D)$.

Proof. Follows easily from the cited Theorem.

□

Lemma 31. [CL, 5.2] *Let X,Y be proper of relative dimension 2, over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K and $f : X \to Y$ a birational map which is an isomorphism in codimension one. Let $\mathcal{C} \subset Div^{eff}_{\mathbb{R}}(X)$ be a cone, and fix a geometric valuation Γ of X. Then the asymptotic order of vanishing σ_Γ is linear on $\mathcal{C}$ if and only if it is linear on $f_*\mathcal{C} \subset Div^{eff}_{\mathbb{R}}(Y)$.*

Proof. As in the cited theorem, substituting Lemma 30 as necessary. ☐

11. Contractions given by Extremal Rays / R

Lemma 32. [CL, 6.2]*Let (X, Δ) be a log smooth klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Suppose (X, Δ) is not nef, and A is a big $\mathbb{Q}$-divisor such that $(X, \Delta + A)$ is klt, and $K_X + \Delta + A$ is nef. Let λ be the nef threshold. Then $\lambda \in \mathbb{Q}^+$, and there is $R \subset \overline{NE}(X)$ with $(K_X + \Delta + \lambda A).R = 0$ and $(K_X + \Delta).R < 0$.*

Proof. As in the cited theorem, but substituting Theorems 27 and 21 for their corresponding versions. ☐

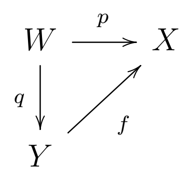
12. Negativity Lemma

Lemma 33. [HK, 1.7] *Let $f : X \to Y$ and $g : X \to Z$ be birational contractions where all spaces considered are schemes of relative dimension 2 over DVR with perfect residue field. Suppose $f^*(A) + E = g^*(B) + F$ for A ample, B nef, E f-exceptional, and F g-exceptional all divisors extending to both fibers. Then $f \circ g^{-1} : Z \to Y$ is regular.*

Proof. It suffices to apply negativity lemma (c.f. [Xu, 2.1] for positive characteristic and $dim \leq 3$) to the special fiber so that $E = F$ and thus $f^*(A) = g^*(B)$. The result then follows from [Ke, 1.0]. ☐

Lemma 34. [CL, 6.4] *Let (X, Δ) a projective klt pair, and $f : X \to Y$ a composite of $(K_X + \Delta)$-divisorial contractions and $(K_X + \Delta)$-flips.*

Then for every resolution



of f*,*

$$p^*\left(K_X+\Delta\right)=q^*\left(K_Y+f_*\Delta\right)+E$$

with $E>0$ *a* q*-exceptional divisor. Therefore* f *cannot be an isomorphism. By the above formula,*

$$H^0\left(X,K_X+\Delta\right)\approx H^0\left(Y,K_Y+f_*\Delta\right).$$

Proof. Follows easily from [KM, 3.38]. □

Lemma 35. [Har, Exc II.4.2] *Let* X *be a reduced scheme,* Y *a separated scheme, and let* f *and* g *be two morphisms from* X *to* Y*. Assume that* $f|_U=g|_U$ *on a Zariski dense open subset* $U\subset X$*. Then* $f=g$*.*

13. Flips

Definition 36. [KM, 3.33] Let X a nomal scheme and D a $\mathbb{Q}$-divisor on X such that K_X+D is $\mathbb{Q}$-Cartier. A (K_X+D)**-flipping contraction** is a proper birational morphism $f:X\to Y$ to a normal scheme Y such that $Exc(f)$ has codimension at least two in X and $-(K_X+D)$ is f-ample. A normal scheme X^+ together with a proper birational morphism $f^+:X^+\to Y$ is called a $(K+D)$-**flip** of f if

(1) $K_{X^+}+D^+$ is $\mathbb{Q}$-Cartier, where D^+ is the birational transform of D on X^+

(2) $K_{X^+} + D^+$ is f^+-ample, and

(3) $Ex\,(f^+)$ has codimension at least two in X^+.

The induced rational map $\phi : X \to X^+$ is sometimes called a $(K + D)$-flip by abuse of notation.

Theorem 37. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt big pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Suppose that $f : X \to Y$ is a $K + \Delta$ flipping contraction. If $R\,(K_X + \Delta)$ is finitely generated, then the $(K + \Delta)$-flip of f exists.*

Proof. Follows easily as in [KM, 6.2], [CL, 6.3]. □

Remark 38. By [JKW, 1.2], the flip of f in the situation above is $\mathbb{Q}-$factorial.

14. Running the Minimal Model Program for General Type

Theorem 39. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair of relative dimension 2 over the spectrum of a DVR $S = Spec\ R$ where R has perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Suppose Δ is big and log smooth over R. Then the minimal model program for (X, Δ) can be run, resulting in a (possibly non-terminating) sequence of flips and divisorial contractions.*

Proof. If necessary, extend R to be complete with algebraically closed residue field k. I begin as in [CL, 6.2]. Let A be a big divisor such that

$K_X + \Delta + A$ is nef. Let α_1 be the smallest positive real number such that $K_X + \Delta + \alpha_1 A$ is nef. Denote by $\varphi : Div_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection, and let $\|\cdot\|$ any norm on $N^1(X)_{\mathbb{R}}$. Pick finitely many big $\mathbb{Q}$-divisors $H^1, ..., H^r$ such that:

(1) $\|\varphi(\Delta + \alpha_1 A) - \varphi(H^i)\| \ll 1$ for all i

(2) writing $\mathcal{C} = \sum_{i=1}^r \mathbb{R}_+(K_X + H^i) \subset Div_{\mathbb{R}}(X)$, we have $K_X + \Delta + \alpha_1 A \in int\ \mathcal{C}$, and the dimension of the cone $\varphi(\mathcal{C}) \subset N^1(X)_{\mathbb{R}}$ is $dim\ N^1(X)_{\mathbb{R}}$

(3) (X, H_i) is klt and log smooth for all i with $supp(K_X + \sum H_i)$ simple normal crossings.

In the first step, since H_i are log smooth, then by Theorem 21, the ring

$$R\left(X, K_X + \Delta, K_X + H^1, ..., K_X + H^r\right)$$

is finitely generated. Thus applying Theorem 27, $Supp\ R = \bigcup C_j$ is rational polyhedral and $C_j = Supp\ R \cap \varphi^{-1}(Nef(X))$ is rational polyhedral, and there is a rational codimension 1 face of $Nef(X)$. Thus $\alpha_1 \in \mathbb{Q}_+$, and we can find an extremal ray $\mathfrak{R} \subset \overline{NE}(X)$ dual to this face which will satisfy $(K_X + \Delta + \alpha_1 A).\mathfrak{R} = 0$ and $(K_X + \Delta).\mathfrak{R} < 0$. By Theorem 29, this ray can be contracted under a birational contraction: $f' : X \to X'$.

If f' is a divisorial contraction, write $f_1 = f'$ and set $\Delta_2 = f_{1*}\Delta$, $A_2 = f_{1*}A$, so A_2 is again big and $K_{X_2} + \Delta_2 + \alpha_1 A_2$ is nef. By properties 1 and 3 above we have finite generation of $R(X_2, K_{X_2} + \Delta_2)$,

$R\left(X_2, K_{X_2}+\Delta_2, K_{X_2}+\Delta_2+A_2\right)$, and

$$R\left(X_2, K_{X_2}+\Delta_2, K_X+H_2^1, ..., K_X+H_2^r\right)$$

$$\approx R\left(X, K_X+\Delta, K_X+H^1, ..., K_X+H^r\right)$$

so the argument in the previous paragraph gives another extremal ray and the process can be repeated.

If, on the other hand, f' is a small contraction, then by Theorem 37, and applying Theorem 18, the flip of the contraction exists (c.f. definition 36). Write f_1 for the induced morphism to the flip and set $\Delta_2 = f_{1*}\Delta$, $A_2 = f_{1*}A$, so A_2 is again big and $K_{X_2}+\Delta_2+\alpha_1 A_2$ is nef. By Lemma 30, we have finite generation of

$$R\left(X_2, K_{X_2}+\Delta_2\right),$$

$$R\left(X_2, K_{X_2}+\Delta_2, K_{X_2}+\Delta_2+A_2\right)$$

and by Lemma 30 we can let $H_2^r = f_{1*}H^r$, and by finite generation of

$$R\left(X_2, K_{X_2}+\Delta_2, K_{X_2}+H_2^1, ..., K_{X_2}+H_2^r\right)$$

we can similarly repeat the process. □

The next theorem assumes we have arrived at a sequence of flips.

Theorem 40. *Notation as above, there is no infinite sequence of flips starting at a given step j of the minimal model program described in Theorem 39. Thus in this case, the minimal model program with scaling ends in a minimal model $(X_g, \Delta_g) = (X_{j'}, \Delta_{j'})$ on which all of the* *following rings are finitely generated:* $R\left(K_{X_{j'}}+\Delta_{j'}\right)$,

$$R\left(K_{X_{j'}}+\Delta_{j'},K_{X_{j'}}+\Delta_{j'}+A_{j'}\right),$$

and

$$R_{j'}=R\left(K_{X_{j'}}+\Delta_{j'},K_{X_{j'}}+H_{j'}^{1},...,K_{X_{j'}}+H_{j'}^{r}\right).$$

Proof. If necessary, extend R to be complete with algebraically closed residue field k. Suppose to the contrary there is an infinite sequence of flips starting at step j. As in Theorem 39, we have for the j^{th} step, finitely many big $\mathbb{Q}$-divisors $H_1^j,...,H_r^j$ such that:

(1) $\|\varphi\left(\Delta_j+\alpha_jA_j\right)-\varphi\left(H_j^i\right)\|\ll 1$ for all i

(2) writing $\mathcal{C}^j=\mathbb{R}_+\left(K_{X_j}+\Delta_j\right)+\sum_{i=1}^{r}\mathbb{R}_+\left(K_{X_j}+H_j^i\right)\subset Div_{\mathbb{R}}\left(X_j\right)$, we have $K_{X_j}+\Delta_j+\alpha_jA_j\in int\ \mathcal{C}_j$, and the dimension of the cone $\varphi\left(\mathcal{C}^j\right)\subset N^1\left(X_j\right)_{\mathbb{R}}$ is $dim\ N^1\left(X_j\right)_{\mathbb{R}}$

(3) $\left(X_j,H_j^i\right)$ are Kawamata Log Terminal

(4)

$$R\left(K_{X_j}+\Delta_j\right),$$

$$R\left(K_{X_j}+\Delta_j,K_{X_j}+\Delta_j+A_j\right),$$

and

$$R_j=R\left(K_{X_j}+\Delta_j,K_{X_j}+H_j^1,...,K_{X_j}+H_j^r\right)$$

are all finitely generated.

(5) $K_{X_j}+\Delta_j+\alpha_jA_j$ is nef.

I proceed almost verbatim as in [CL, 6.5]. By construction, C^j contains an open neighborhood of the nef divisor $K_{X_j}+\Delta_j+\alpha_jA_j$, so C^j

contains ample divisors in its interior. Thus the cone $\varphi(Supp\ R_j) \subset N^1(X)$ also has dimension $dim\ N^1(X)_{\mathbb{R}}$. Let $Supp\ R = \bigcup_k C_k^j$ the coarsest finite rational polyhedral subdivision from Theorem 24(3). For $i > j$, let $C_k^i \subset Div_{\mathbb{R}}(X_i)$ denote the proper transform of C_k^j and $C^i \subset Div_{\mathbb{R}}(X_i)$, the proper transform of C^j. By Lemma 31, for every geometric valuation Γ, the asymptotic order function σ_Γ is linear on each C_k^j.

By construction if $0 < \alpha \leq \alpha_1$, then $K_{X_j} + \Delta_j + \alpha A_j \in Int\ C^j$, so $K_{X_i} + \Delta_i + \alpha_i A_i \in int\ C^i$ for all $i > j$. Since $K_{X_i} + \Delta_i + \alpha_i A_i$ is nef, then by requirement (4) above and Lemma 25, $K_{X_i} + \Delta_i + \alpha_i A_i \in Supp\ R_i$. Thus by Theorem 27, for each i there exists an index k such that the image of C_k^i in $N^1(X_i)_{\mathbb{R}}$ is a subset of Nef(X_i). Therefore,

$$\varphi\left(C_k^1\right) \subset (f_{i-1} \circ \cdots \circ f_1)^* \, Nef\,(X_i)\,.$$

Since there are finitely many cones C_k^j, there are two indices p and q such that $(f_{p-1} \circ \cdots \circ f_1)^* \, Nef\,(X_p)$ and $(f_{q-1} \circ \cdots \circ f_1)^* \, Nef\,(X_q)$ share a common interior point. Thus, by Lemma 33, the map $X_p \to X_q$ is a morphism. By Lemma 35, it is therefore an isomorphism. This contradicts Remark 38. □

15. Reductions To General Case

Lemma 41. [HMX2014, 2.8.3] *Let (X, Δ) be a log smooth pair which is a scheme of dimension at most 3, with the coefficients of Δ belonging to $(0, 1]$, and with X projective. If (X, Δ) has a weak log canonical model, then there is a sequence $\pi : Y \to X$ of smooth blow ups of the* *strata of* Δ *such that if we write* $K_Y + \Gamma = \pi^* (K_X + \Delta) + E$*, where* $\Gamma \geq 0$ *and* $E \geq 0$ *have no common components,* $\pi_* \Gamma = \Delta$ *and* $\pi_* E = 0$ *and if we write*

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma (Y, K_Y + \Gamma),$$

then $\mathbb{B}_- (Y, K_Y + \Gamma')$ *contains no strata of* Γ'*. If* Δ *is a* $\mathbb{Q}$*-divisor, then* Γ' *is a* $\mathbb{Q}$*-divisor.*

Proof. Follows easily from the cited theorem. □

Lemma 42. [HMX2014, 5.3]*Let* X/R *a projective scheme of relative dimension* 2 *over a DVR* R*, with algebraically closed residue field and perfect fraction field. Let* (X, Δ) *be a log smooth dlt pair with* X $\mathbb{Q}$*-factorial and projective and* Δ *a* $\mathbb{Q}$*-divisor. If* Φ *is a* $\mathbb{Q}$*-divisor such that*

$$0 \leq \Delta - \Phi \leq N_\sigma (X, K_X + \Delta),$$

then steps of the $K_X + \Phi$*-minimal model program are steps of the* $K_X + \Delta$ *minimal model program, and so termination for* (X, Φ) *implies termination for* (X, Δ)*.*

Proof. (Similar proof to the cited Theorem). Let $f : X \to Y$ the result of running a $(K_X + \Phi)$ minimal model program and let $Y \to Z$ be the ample model of $K_X + \Phi$. Let $\Delta_t = t\Delta + (1 - t) \Phi$ so that if $0 < t \ll 0$, then f is also a $(K_X + \Delta_t)$-MMP. If C generates a $K_X + \Delta_t$-negative extremal ray, suppose that $(K_X + \Phi) .C > 0$. Applying Theorem 10 on the special fiber,

$$-4 \leq (K_X + \Delta) .C < 0.$$

Then, following the inequalities in [HMX2014, 5.1], gives a contradiction if $0 < t < \frac{1}{5}$. Thus every step of the $K_X + \Delta_t$ with scaling of an ample divisor is $(K_X + \Phi)$-trivial, so after finitely many steps there is a model $g : X \to W$ contracting the components of $N_\sigma (X, K_X + \Delta_t)$ (this holds with no changes from [HMX2014, 2.7.1]). Thus, as $\mathit{supp}\,(\Delta - \Phi) \subset \mathit{supp}\, N_\sigma (X, K_X + \Delta) = \mathit{supp}\, N_\sigma (X, K_X + \Delta_t)$, then $g_* (K_Y + \Phi)$ is semiample whenever $g_* (K_X + \Delta)$ is, and [HMX2014, 2.7.2] (which holds using only the log smooth resolution) implies that g is a minimal model of (X, Δ), which is good when (X, Φ) has a good minimal model. □

Lemma 43. [GL, 2.3] *Let X/R a scheme which is projective of relative dimension 2 over either a DVR R with algebraically closed residue field and perfect fraction field or over a field. Let (X, Δ) be a log smooth klt pair and $\varphi : W \to X$ a log resolution of (X, Δ). Choose Δ_W so that $K_W + \Delta_W = \varphi^* (K_X + \Delta) + E$ with Δ_W and E effective $\mathbb{Q}$-Weil divisors with no common component. Let $F = \sum_{F_i\ \varphi-exceptional\ prime\ divisor} F_i$ and $\Delta_W^\epsilon = \Delta_W + \epsilon F$, then (W, Δ_W^ϵ) is an ϵ-log smooth model of (X, Δ). Then for $0 < \epsilon \ll 1$, a (good) minimal model $\phi : (W, \Delta_W^\epsilon) \to \left(W_{min}, \Delta_{W_{min}}^\epsilon\right)$ is also a (good) minimal model for (X, Δ).*

Proof. Follows easily from [BCHM, 3.6.10, 3.6.11]. □

16. Existence of Minimal Models Special Case

Lemma 44. [HMX2010, 3.1] *Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k). Let $f : X \to Y$ be a step of the $(K_X + \Delta)$-MMP. If f is birational and V is a non-canonical centre of (X, Δ), then V is not contained in the indeterminacy locus of f, V_0 is not contained in the indeterminacy locus of f_0, and the induced maps $\phi : V \to W$ and $\phi_k : V_k \to W_k$ are birational, where $W = f(V)$. Let $\Gamma = f_*\Delta$. Then $\mathbb{B}_-(Y_0, K_{Y_0} + \Gamma_0)$ contains no non-canonical centres of (Y_0, Γ_0) (so we may repeat the process), and if V is a non-klt centre or $V = X$, then $\phi : V \to W$ and $\phi_0 : V_0 \to W_0$ are birational contractions. If f is a Mori fibre space, then f_0 is not birational.*

Proof. (Follows easily from the cited Theorem). Suppose f is birational. Let V be a non-canonical centre of (X, Δ). Let $g : X \to Z$ be the contraction of the extremal ray associated to f (so that $f = g$ unless f is a flip). Let $Q = g(V)$, and let $\psi : V \to Q$, be the induced morphism. As every component of V_k is a non-canonical centre of (X_k, Δ_k), then by hypothesis, components of V_k are not contained in $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$, thus ψ_k is defined at each such component, hence is birational. By upper-semicontinuity of the fibers of ψ, ψ is birational, and thus $\phi : V \to Q \to W$, and ϕ_k are both birational.

Now suppose V is a non-klt centre or $V = X$, the above holds in the first case, as non-klt centres are non-canonical. Comparing the discrepancies of the differents of adjunction for Δ_k and Γ_k as in the cited proof shows that ϕ_k, f_k, and thus ϕ are birational contractions. On the other hand, if f is a Mori fibre space, then, as the dimension of the fibers of $f : X \to Y$ are upper-semicontinuous, f_k is not birational. $\square$

Theorem 45. [HMX2014, 3.2] *Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $\mathbb{B}_- (X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k) and is log smooth Let $f : X \to Y$ be a step of the $(K_X + \Delta)$-MMP. Then the minimal model program with scaling terminates using only contractions.*

Proof. (Similar proof to the cited theorem). Let $f : X \to Y$ the $(K_X + \Delta)$-MMP with scaling of an ample divisor A. Let $\Gamma = f_*\Delta$ and $B = f_*A$. By construction, $K_Y + tB + \Gamma$ is nef for some $t > 0$. By Lemma 44, $f : X \to Y$ is a birational contraction and $f_k : X_k \to Y_k$ is a birational contractions from $(X_k, tA_k + \Delta_k)$. If $K_X + \Delta$ is not pseudo-effective, then for $t > 0$, the result of f is a Mori fiber space, and by Lemma 44, Y_k is covered by $K_{Y_k} + tB_k + \Gamma_k$-negative curves, contradicting bigness of $K_{X_k} + tA_k + \Delta_k$. Thus $K_X + \Delta$ is pseudo-effective, and given any $\epsilon > 0$, we may run the MMP until $t < \epsilon$. Now we conclude by Theorem 22. Letting ϵ be the constant given in that theorem, the minimal models for $(X_k, tA_k + \Delta_k)$ are all isomorphic $t \in [0, \epsilon]$. Thus, once $t < \epsilon$, any more steps in the minimal model program with scaling must be an isomorphism on the special fiber, and thus an isomorphism. Thus there exists a minimal model (Y, Γ) for (X, Δ). $\square$

17. Existence of Minimal Models General Case

Theorem 46. *Let* (X, Δ) *be a klt pair of relative dimension* 2*, proper over a DVR* R *with perfect residue field* k *of characteristic* $p > 0$ *and perfect fraction field* K*. Assume that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier, and log smooth over* R*. Then the minimal model of* (X, Δ) *exists.*

Proof. I again start by repeating the reduction of [HMX2014, 6.1]. If necessary, extend R to be complete with algebraically closed residue field k, so the strata of Δ have irreducible fibers over R. Let $f_0 : Y_0 \to X_0$ be the birational morphism of Lemma 41. Under the log smooth hypothesis, and since the strata of Δ have irreducible fibers, and f_0 blows up strata of Δ_0, extend f_0 to a birational morphism $f : Y \to X/R$ which is a composition of smooth blow ups of strata of Δ. Write

$$K_Y + \Gamma = f^* (K_X + \Delta) + E$$

with $\Gamma \geq 0$ and $E \geq 0$, $f_*\Gamma = \Delta$, and $f_*E = 0$. Then (Y, Γ) is log smooth and the fibres of components of Γ are irreducible. By Lemma 43, (X, Δ) has a minimal model if (Y, Γ_ϵ) has a good minimal model, where $(Y, \Gamma_\epsilon) = (Y, \Gamma + \epsilon F)$ is the ϵ-log smooth model of (X, Δ) with F the sum of f-exceptional divisors and $0 < \epsilon \ll 1$.

Replace (X, Δ) by (Y, Γ_ϵ) and set $\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma (X_k, K_{X_k} + \Delta_k)$ so that $\mathbb{B}_- (X_k, K_{X_k} + \Theta_k)$ contains no strata of Θ_k. Let $0 \leq \Theta \leq \Delta$ be the unique divisor such that $\Theta_k = \Theta|_{X_k}$. Let H relatively ample such that $(X, \Theta + H)$ is log smooth over R, and $K_X + \Theta + H$ is big. There is a commutative diagram:

$$\begin{array}{ccc} \pi_* \mathcal{O}_X \left(m \left(K_X + \Theta \right) + H \right) & \longrightarrow & \pi_* \mathcal{O}_X \left(m \left(K_X + \Delta \right) + H \right) \\ \downarrow & & \downarrow \\ H^0 \left(X_k, \mathcal{O}_{X_k} \left(m \left(K_{X_k} + \Theta_k \right) + H_k \right) \right) & \longrightarrow & H^0 \left(X_k, \mathcal{O}_{X_k} \left(m \left(K_{X_k} + \Delta_k \right) + H_k \right) \right) \end{array}$$

with surjective columns by Theorem 17 and [Liu, 5.3.20(b)], and with the bottom row an isomorphism. Applying Nakayama's lemma gives an isomorphism on the top row, so that $\Theta \geq \Delta - \Delta \wedge N_\sigma \left(X, K_X + \Delta \right)$. Again applying Theorem 17, gives $\Theta = \Delta - \Delta \wedge N_\sigma \left(X, K_X + \Delta \right)$. Thus $\Delta - \Theta \leq N_\sigma \left(X, K_X + \Delta \right)$, so by Lemma 42, it suffices to find a minimal model for (X, Θ). Replacing (X, Δ) by (X, Θ), it suffices to assume that $\mathbb{B}_- \left(X_k, K_{X_k} + \Delta_k \right)$ contains no strata of Δ_0. Letting A be an ample divisor, we run the minimal model program with scaling of A. Now the assumptions of Theorem 45 apply, so we know that (X, Θ) has a minimal model. □

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