

LOG MINIMAL MODELS FOR ARITHMETIC THREEFOLD GERMS

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ABSTRACT. In this note, I attempt to finish the log minimal model program proofs started in [EGSurfDIOP] and [EgSyz]. In the first note, finite generation of the log canonical ring is proven for Kawamata Log Terminal, log smooth pairs of general type on an algebraic space having two dimensional fibers over a DVR. In the [EgSyz], the minimal model program for such pairs is studied. In this note, I study pairs of any Kodaira dimension.

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1. INTERSECTION THEORY ON X/R

Suppose X/R is a proper algebraic space of relative dimension 2 over a discrete valuation ring R with residue field k of characteristic $p > 0$ and perfect fraction field K (See for example the compact Shimura Varieties mentioned in [Suh, 0.1]).

Lemma 1. [KU, 9.3] *Let $\varphi : X \rightarrow \text{Spec}(R)$ be an algebraic two-dimensional space, smooth, proper, separated, and of finite type over $\text{spec}(R)$, where R is a DVR with algebraically closed residue field k , and field of fractions K . Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ . If D, D' are divisors on X , then*

$$\begin{aligned} K_{X_k}^2 &= K_{X_K}^2 \\ (K_{X_k} \cdot D_k)_{X_k} &= (K_{X_K} \cdot D_K)_{X_K} \\ (D_k \cdot D'_k)_{X_k} &= (D_K \cdot D'_K)_{X_K}. \end{aligned}$$

If X/R is smooth, then Lemma 1 applies to show that for any divisors C, D extending to both fibers, we can define $C \cdot D = (C|_{X_k} \cdot D|_{X_k})_{X_k}$. On the other hand, if X/R is merely normal and proper, but is actually a scheme, the resolution of singularities, Theorem 3, holds. Thus the intersection theory can be defined as in [Tan, Def 3.1]: $f : X' \rightarrow X$ is a resolution, and $C \cdot D = f^*C \cdot f^*D$ for two divisors C, D on Y/R . By properness, this intersection extends by linearity to Weil divisors with

\mathbb{Q} or \mathbb{R} coefficients. Numerical equivalence and $N^1(X)_{\mathbb{Q},\mathbb{R}}$ are then defined as usual.

Note the two different hypothesis here: The invariance of plurigenera theorem requires a proper, smooth algebraic space of relative dimension 2, while the existence of minimal models requires the intermediate steps of the minimal model program to have some intersection theory, and thus be schemes. Recall that a Cartier divisor C on a variety is usually called nef if $C.D \geq 0$ for all curves C .

Lemma 2. *Let S be an affine Dedekind scheme and $f : X \rightarrow S$ a projective morphism. Let \mathcal{L} be an invertible sheaf on X such that \mathcal{L}_s is nef for every closed point $s \in S$. Then \mathcal{L} is nef.*

Proof. By [Liu, 5.3.24], the statement holds when nef is replaced by ample. Now, c.f. [Laz, 1.4.10], it suffices to choose an ample divisor H which restricts to X_s , so that $D_s + \epsilon H_s$ is ample for all sufficiently small ϵ . Then $D + \epsilon H$ is ample for all sufficiently small ϵ , and so D is nef. \square

2. LOG RESOLUTION

The following statement of resolution will be used. So for example, if X is an algebraic variety over a number field K then, by [MP, III.2], there is a completed arithmetical variety \overline{X} of dimension $\dim(X) + 1$:

$$\overline{X} \rightarrow \overline{\text{Spec } \mathcal{O}_K}.$$

(Note that a complete DVR is excellent).

Theorem 3. [CP2014, 1.1] *Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \rightarrow X$ with the following properties:*

- (i) X' is everywhere regular
- (ii) π induces an isomorphism $\pi^{-1}(\text{Reg } X) \approx \text{Reg } X$
- (iii) $\pi^{-1}(\text{Sing } X)$ is a strict normal crossings divisor on X' .

The following Statement of Log Resolution will be used in the Log Smooth case (i.e. the case with (X, Δ) a pair with X smooth and Δ simple normal crossings.)

Theorem 4. [CP2014, 4.3] *Let S be a regular Noetherian irreducible scheme of dimension three which is excellent and $\mathcal{I} \subset \mathcal{O}_S$ be a nonzero ideal sheaf. There exists a finite sequence*

$$S =: S(0) \leftarrow S(1) \leftarrow \cdots \leftarrow S(r)$$

with the following properties:

- (i) for each j , $0 \leq j \leq r-1$, $S(j+1)$ is the blowing up along a regular integral subscheme $\mathcal{Y}(j) \subset \mathcal{S}(j)$ with

$$\mathcal{Y}(j) \subseteq \{s_j \in \mathcal{S}(j) : \mathcal{I}\mathcal{O}_{S(j), s_j} \text{ is not locally principal}\}.$$

- (ii) $\mathcal{I}\mathcal{O}_{S(r)}$ is locally principal.

3. EFFECTIVE KLT KAWAMATA VIEHWEG VANISHING

In this section, I note that a simple modification of Tanaka's vanishing theorem [Tan2] gives a version of the (log) Kawamata Viehweg

Vanishing in which the multiple of the nef divisor is at least computable. There is a similar recent result in [DF], which is more restrictive in that it requires a smooth variety rather than a normal projective klt pair (the situation of a pair is considered in this paper). First, a theorem of Terakawa:

Theorem 5. [Ter] *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X . Assume that either*

(1) $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or

(2) X is of general type with

$p \geq 3$ and $(D^2) > \text{vol}(X)$ or

$p = 2$ and $(D^2) > \max\{\text{vol}(X), \text{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all $i > 0$.

Applying the above to Tanaka's proof gives the following:

Theorem 6. (Effective KLT Kawamata-Viehweg Vanishing for normal surfaces) *Let (X, Δ) be a normal projective klt surface over an algebraically closed field of characteristic $p > 2$, where Δ is an effective \mathbb{R} -divisor. Let N be a nef and big \mathbb{R} -cartier \mathbb{R} -divisor with $N^2 > \max\{\text{vol}(X), \text{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$. Let D be a \mathbb{Q} -Cartier \mathbb{Z} -divisor such that $D - (K_X - \Delta)$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(D + rN + N')) = 0$$

for every $i > 0$, every positive real number $r \geq 1$, and every nef \mathbb{R} -Cartier \mathbb{R} -divisor N' such that $rN + N'$ is a Cartier divisor.

Proof. Verbatim from [Tan2, 2.11] except that Weak Kodaira Vanishing [Tan2, 2.4] in the proof of [Tan2, 2.6] is replaced by Terakawa's vanishing Theorem 5. □

4. INVARIANCE OF PLURIGENERA SPECIAL CASE

Lemma 7. [KU, 9.4] *Let $\varphi : X \rightarrow \text{Spec}(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $\text{spec}(R)$, where R is a DVR with algebraically closed residue field k , and field of fractions K . Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ . If X_k contains an exceptional curve of the first kind e , there exists a DVR $\tilde{R} \supset R$, with residue field isomorphic to k , and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \rightarrow \text{Spec}(\tilde{R})$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \rightarrow \tilde{Y}$ over $\text{Spec}(\tilde{R})$ such that on the closed fibre, π induces the contraction of the exceptional curve e . Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.*

Proof. (Follows easily from the cited theorem, although I make a small observation about the extension of residue fields). By [A1, Cor 6.2] $\text{Hilb}_{X/\text{Spec}(R)}$ is represented by an algebraic space \mathcal{H} which is locally of finite type over $\text{Spec}(R)$. Let Y be the irreducible component containing the point $\{e\}$ corresponding to the exceptional curve e on the

special fiber. Then $e \approx \mathbb{P}_k^1$ and $N_{e/X} \approx \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, so Y is regular at $\{e\}$ and of dimension 1. Since e is fixed in the special fiber, the structure morphism $p : Y \rightarrow \text{Spec}(R)$ is surjective. By [Bo, 2.2.14]¹, we can find an étale cover $\tilde{R} \supset R$ and a morphism $j : \text{Spec}(\tilde{R}) \rightarrow Y$ over $\text{Spec}(R)$ with $j(\tilde{o}) = \{e\}$ (if R is not already complete, then first extend R to a complete DVR using [Gr, 0.6.8.2,3] so that \tilde{R} is again a DVR). As k is assumed algebraically closed, and $\tilde{R} \rightarrow R$ is unramified, then the extension of residue fields is finite and separable at the closed point of R , and hence an isomorphism of residue fields. Let $\hat{p} : \mathcal{E} \rightarrow \text{Spec}(\tilde{R})$ be the pull-back of the universal family over Y . As the closed fibre \mathcal{E}_0 is a projective line, we may choose the morphism j in such a way that the generic fibre \mathcal{E}_1 of \hat{p} is also a projective line. Moreover, \mathcal{E} can be considered as a smooth closed algebraic subspace of codimension 1 in $\hat{X} = X \otimes \tilde{R}$. By Lemma 1, $-1 = \mathcal{E}_0^2 = \mathcal{E}_1^2$. Hence \mathcal{E}_1 is also an exceptional curve of the first kind. Hence by [A2, Cor 6.11] there exists a contraction morphism $\pi : \hat{X} \rightarrow \hat{X}$ over $\text{Spec}(\tilde{R})$ which contracts \mathcal{E} to a section of $\tilde{\varphi} : \tilde{X} \rightarrow \text{Spec}(R)$ where $\tilde{\varphi}$ is proper, smooth, separated and of finite type over $\text{Spec}(\tilde{R})$.

□

Lemma 8. [KK94, 2.3.5] *Let (S, B) be a log-canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C \cdot (K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

¹If k is perfect, the proof of the cited lemma seems to go through when $f : X \rightarrow S$ is merely smooth at $x \in X$ and $f : X \rightarrow S$ is locally of finite type.

Theorem 9. [Tan, 4.4, 4.6] *Let X be a projective normal surface and let Δ be an effective \mathbb{R} -divisor such that $K_X + \Delta$ is \mathbb{R} -cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let H be an \mathbb{R} -cartier ample \mathbb{R} -divisor. Then the following assertions hold:*

- (1) $\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0} [C_i]$
- (2) $\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta + H \geq 0} + \sum_{finite} \mathbb{R}_{\geq 0} [C_i]$
- (3) *Each C_i in (1) and (2) is rational or $C_i = B_j$ for some B_j with $B_j^2 < 0$.*
- (4) *Each C_i in (1) and (2) satisfies $0 < -C_i \cdot (K_X + \Delta) \leq 3$.*

Lemma 10. [Tan, 5.3] *Let X be a projective normal surface and let C be a curve in X such that $r(K_X + C)$ is Cartier for some positive integer r .*

- (1) *If $C \cdot (K_X + C) < 0$, then $C \approx \mathbb{P}^1$.*
- (2) *If $C \cdot (K_X + C) = 0$, then $C \approx \mathbb{P}^1$ or $\mathcal{O}_C \left((K_X + C)^{[r]} \right) \approx \mathcal{O}_C$.*

Proposition 11. *Let (X, Δ) be a terminal log smooth pair of relative dimension 2 over a DVR R with algebraically closed residue field k of characteristic $p > 2$ and perfect fraction field K . Assume that $K_X + \Delta$ is \mathbb{Q} -Cartier and $\kappa(K_{X_k} + \Delta_k) = 2$. Assume that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$. Then there exists a computable m_0 such that for $m > m_0$,*

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

The proof of proposition 11 is given in the following claims.

Claim. Assumptions as above, after passing to an extension R' of R , there is a proper, smooth algebraic space X^{min}/R' and an R' morphism

$X \otimes R' \rightarrow X^{min}$ such that both $X_k \rightarrow X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef.

Proof. As k is algebraically closed of characteristic $p > 0$, then by the Cone Theorem 9,

$$\overline{NE}(X_k) = \overline{NE}(X_k)_{K_{X_k} + \Delta_k \geq 0} + \sum \mathbb{R}_{\geq 0} [C_i]$$

with each C_i is rational or $C_i = B_j$ for some B_j a component of Δ with $B_j^2 < 0$. Under the assumption that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$, then actually each C_i is rational and is not a component of Δ_k . Thus $C_i \cdot \Delta_k \geq 0$ so $C_i \cdot K_{X_k} < 0$, and if $C_i^2 > 0$, then by [KK94, 2.3.3], then $-(K_{X_k} + \Delta_k)$ is ample, which contradicts the assumption that $\kappa(K_{X_k} + \Delta_k) = 2$. Thus Theorem 8 implies $C_i \approx \mathbb{P}^1$ and C_i can be contracted to a log-canonical point, which is actually a smooth point under the terminal assumption. Thus C_i is an exceptional curve of the first kind, so it is possible to apply Lemma 7.

By Lemma 7, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \rightarrow \tilde{X}$ induces the contraction of C_i on X_k and X_K , and further $\tilde{\varphi} : \tilde{X} \rightarrow \text{Spec}(\tilde{R})$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since k is algebraically closed, it induces identity on residue fields. Now I need to work with \tilde{X} . \tilde{X}_k is projective, so the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no

problem in extending R . As the Picard number of the special fiber drops each step, there are only finitely many steps. \square

Claim 12. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef.

Proof. It suffices to apply Lemma 2 to X_k^{min} , and then restrict to X_K . \square

Now we proceed by cases depending on the Kodaira dimension.

Claim 13. There is a computable m_0 such that for $m_0|m$, we have

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

Proof. Suppose first that (X_k, Δ_k) is big. By the above, we achieve X^{min} such that $K_X + \Delta$ is nef on both X_k^{min} and X_K^{min} . By Theorem 6 applied to the special fiber (and the semicontinuity theorem) there exists a computable $m_0 \gg 0$ such that for $m_0 < m$ and $i > 0$, we have

$$h^i(m(K_{X_K} + \Delta_K)) = h^i(m(K_{X_k} + \Delta_k)) = 0.$$

In fact, if $\Delta = 0$, then Ekedahl's vanishing Theorem [Ek] can be applied and $m = 2$. Thus, applying the invariance of Euler characteristic [Liu, 5.3.22], and birational invariance of the plurigenera, it follows that for $m > m_0$,

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

\square

5. INVARIANCE OF PLURIGENERA AND FINITE GENERATION

Theorem 14. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $K_X + \Delta$ is big, \mathbb{Q} -Cartier, and simple normal crossings over R . Then, there exists a (computable) m_0 depending on the intersection numbers such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

Proof. Since the hypothesis and conclusion are preserved by base change, then after extending R if necessary, we may assume that k is algebraically closed and that R is complete. The proof follows similarly to [HMX2010, 1.6] except that Proposition 11 is used in place of Kawamata Viehweg Vanishing. Recall that the log-canonical ring of $K_{X_k} + \Delta_k$ is finitely generated (c.f. [Tan, 7.1]) so $N_\sigma(K_{X_k} + \Delta_k)$ is a \mathbb{Q} -divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma(K_{X_k} + \Delta_k)$$

and let $0 \leq \Theta \leq \Delta$ be a \mathbb{Q} -divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. Replacing (X, Δ) by a blow-up, assume (X, Δ) is terminal. Letting $m \gg 0$ to fit the hypothesis of Proposition 11, and sufficiently divisible such that $m(K_{X_k} + \Theta_k)$ is integral, then by definition of N_σ ,

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_k} + \Theta_k)).$$

Furthermore, by Proposition 11,

$$h^0(m(K_{X_k} + \Theta_k)) = h^0(m(K_{X_K} + \Theta_K)).$$

As $\Theta \leq \Delta$, the first part of the theorem follows by semicontinuity. \square

6. FINITE GENERATION (BIG CASE)

Corollary 15. *Situation as above, after possibly extending R , the canonical ring $R(K_X + \Delta)$ is finitely generated over R . If $K_X + \Delta$ is pseudo-effective (not necessarily big), a contraction of a $K_{X_k} + \Delta_k$ -negative extremal ray cut out by an ample H over R induces a contraction of a $K_X + \Delta$ negative extremal ray. If the contraction is a flipping contraction, then the flip exists.*

Proof. By [Tan], there is $m \gg 0$, such that

$$\bigoplus_{m \geq 0} H^0(ml(K_{X_k} + \Delta_k))$$

is generated in degree 1, so that there is a surjection

$$S^l H^0(m(K_{X_k} + \Delta_k)) \twoheadrightarrow H^0(ml(K_{X_k} + \Delta_k)).$$

Consider the following diagram:

$$\begin{array}{ccc} S^l H^0(m(K_X + \Delta)/R) & \xrightarrow{\alpha} & H^0(ml(K_X + \Delta)/R) \\ \downarrow & & \downarrow \\ S^l H^0(m(K_{X_k} + \Delta_k)) & \twoheadrightarrow & H^0(ml(K_{X_k} + \Delta_k)) \end{array}$$

The vertical maps are surjective by the first part of the theorem (and applying Nakayama's lemma and cohomology and base change to pull back the generators of the k -modules). Now applying the version of Nakayama's Lemma given in [GH, Chapter 5.3], surjectivity of the

bottom map implies surjectivity of α , and thus $R(K_X + \Delta)$ is finitely generated over R .

Finally, suppose that there is a contraction of a curve on the closed fiber (X_k, Δ_k) corresponding to a $K_{X_k} + \Delta_k$ -negative extremal ray Σ_k . Let $H_k = H|_{X_k}$ be such that $(K_{X_k} + \Delta_k + H_k) \cdot \Sigma_k = 0$. The goal is to show that

$$R(K_X + \Delta + H/R)$$

is finitely generated so that there exists a contraction defined by $X_1 = \text{Proj } R(K_X + \Delta + H/R)$. However, as H is ample, then under the assumption that $K_{X_k} + \Delta_k$ is pseudo-effective, $K_{X_k} + \Delta_k + H_k$ is actually big, and so this follows. \square

Corollary 16. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $K_X + \Delta$ is \mathbb{Q} -Cartier, and simple normal crossings over R . If A_1, \dots, A_m are any ample divisors, then the ring $R(X, K_X + \Delta + A_1, \dots, K_X + \Delta + A_n)$ is finitely generated over R .*

Proof. C.f. the proof of [CZ, 2.11], it suffices to prove that if $G = \sum_{i=1}^k m_i (a(K_X + \Delta_i + A_i))$, with fixed $a \in \mathbb{N}$, $m_1, \dots, m_k \geq 0$, and $l \in \{1, \dots, l\}$ then there exists m_l such that the following maps are surjective

$$\begin{aligned} H^0(X, \mathcal{O}_X(G - a(K_X + \Delta - A_l))) \otimes H^0(X, \mathcal{O}_X(a(K_X + \Delta + A_l))) \\ \rightarrow H^0(X, \mathcal{O}_X(G)). \end{aligned}$$

By applying Nakayama's Lemma and Theorem 14 as in the proof of Theorem 15, it suffices to show the surjectivity on the central fiber.

Applying [Tan], there exists a minimal model (X'_k, Δ'_k) for (X_k, Δ_k) such that $K_{X'_k} + \Delta'_k$ is nef and semi-ample. Thus there is $m \gg 0$ such that $m(K_{X'_k} + \Delta'_k)$ is both base-point free, and $m - 2$ is large enough so that Theorem 6 applies. Now the proof of [CZ, 2.7] applies to give the desired surjectivity. \square

7. FINITENESS OF MODELS FOR SURFACES

Theorem 17. *Let X be a two dimensional normal variety over a perfect field of characteristic $p > 0$. Let (X, Δ) be a log smooth KLT pair and A an ample \mathbb{Q} -divisor on X with a good minimal model (which exists for non-negative Kodaira dimensions by [Tan]). Then there exists an $\epsilon > 0$ such that the minimal models (and the output of the minimal model program with scaling) for $(X, \Delta + tA)$ are all isomorphic for $t \in [0, \epsilon]$.*

Proof. (Essentially same as [Lai, 26,27] and simplifies due to the fact that a birational isomorphism in codimension 1 on a surface gives an isomorphism). Let $\phi : (X, \Delta) \rightarrow (X_g, \Delta_g)$ be a good minimal model. Let $f : X_g \rightarrow Z = Proj R(K_{X_g} + \Delta_g)$ the contraction (which exists by since $K_{X_g} + \Delta_g$ is nef, hence semiample by [Tan]). Then ϕ contracts the divisorial part of $\mathbb{B}(K_X + \Delta)$. Pick $t_0 > 0$ such that $(X_g, \Delta_g + t_0\phi_*A) = (X_g, \Delta_g + t_0A_g)$ is klt with A ample on X (note A_g is big, not in general nef). For H ample on X_g , let

$\phi : X_g \rightarrow X'$ be the minimal model program with scaling, which terminates (again by [Tan]), and gives a minimal model of $(X_g, \Delta_g + t_0\Delta_g)$ over Z . For any curve contracted by f , $(K_{X_g} + \Delta_g) \cdot C = 0$, hence $K_{X'} + \psi_*\Delta_g = K_{X'} + \Delta' \equiv_Z 0$. Thus curves contracted by ψ have trivial intersection with $K_{X_g} + \Delta_g$, and intersect negatively with A_g . Thus changing t does not affect which curves intersect negatively with $K_{X_g} + \Delta_g + tA_g$, and so X' is a minimal model of $(X_g, \Delta_g + tA_g)$ for all $t \in (0, t_0]$.

Now $\Delta' + t_0A'$ with $A' = \psi_*A_g$ (which is big) implies by the Theorem 9, that there exist only finitely many $K_{X'} + \Delta' + t_0A'$ negative extremal rays in $\overline{NE}(X')$. These are all necessarily just intersecting A' negatively, so decreasing t_0 , eventually we get to a point where a further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (Now we are on X' , not over Z). Again shrinking t_0 , suppose that $\psi \circ \phi$ is discrepancy negative w'r't $(X, \Delta + tA)$ for all t in the half open interval $(0, t_0]$. Note that $\mathbb{B}(K_X + \Delta + t_0A) \subset \mathbb{B}(K_X + \Delta)$, so ψ contracts only the curves not contracted by ϕ . Thus $\psi \circ \phi$ is discrepancy negative on the closed interval $[0, t_0]$, so X' is a minimal model of $(X, \Delta + tA)$ for all $t \in [0, t_0]$. Thus $\mathbb{B}(K_X + \Delta + tA)$ has the same divisorial components for all $t \in [0, t_0]$. Hence, any two minimal models for different $t \in [0, t_0]$ are birational and isomorphic in codimension one, and as the total dimension is two, they are thus isomorphic. \square

8. CONE, RATIONALITY, AND CONTRACTION / R

Theorem 18. [CL, 3.5] *Let X be a normal scheme of relative dimension 2, proper over S , the spectrum of a DVR which has perfect*

residue field k of characteristic $p > 0$ and perfect fraction field K . Let D_1, \dots, D_r be \mathbb{Q} -Cartier divisors on X . Assume that $R(X; D_1, \dots, D_r)$ is finitely generated, and let $D : \mathbb{R}^r \ni (\lambda_1, \dots, \lambda_r) \mapsto \sum \lambda_i D_i \in \text{Div}_{\mathbb{R}}(X)$ be the tautological map.

(1) The support of R is a rational polyhedral cone.

(2) If $\text{Supp } R$ contains a big divisor and $D \in \sum \mathbb{R}_+ D_i$ is pseudoeffective, then $D \in \text{Supp } R$.

(3) There is a finite rational polyhedral subdivision $\text{Supp } R = \bigcup C_i$ such that σ_{Γ} is a linear function on C_i for every geometric valuation Γ of X . Furthermore, there is a coarsest subdivision with this property, in the sense that, if i and j are distinct, there is at least one geometric valuation Γ of X such that (the linear extensions of) $\sigma_{\Gamma}|_{C_i}$ and $\sigma_{\Gamma}|_{C_j}$ are different.

(4) There is a finite index subgroup $\mathbb{L} \subset \mathbb{Z}^r$ such that for all $n \in \mathbb{N}^r \cap \mathbb{L}$, if $D(n) \in \text{Supp } R$, then $\sigma_{\Gamma}(D(n)) = \text{mult}_{\Gamma}|D(n)|$.

Proof. (1, 2) follow easily from the cited theorem. For 3, 4 note that [ELM, 4.7] is stated for an arbitrary Noetherian scheme, and this theorem is the only result used in the proof of this result which is given in [CL, 3.5]. \square

Theorem 19. ([CL, 3.6]) *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $K_X + \Delta$ is \mathbb{Q} -Cartier, and simple normal crossings over R . If $K_X + \Delta$ is pseudoeffective, then it is \mathbb{Q} -effective.*

Proof. As in the source paper, but using Theorem 16 for the finite generation. \square

Definition 20. [CL, def 4.1] Let W a finite dimensional real vector space, $C \subset W$ a closed convex cone spanning W , and $v \in W$. The visible boundary of C from v is

$$V = \{w \in \partial C \mid [v, w] \cap C = \{w\}\}.$$

Corollary 21. [CL, 4.2] (*Kawamata's Rationality, Cone and Contraction Theorem*) Let (X, Δ) be a klt pair of relative dimension 2, proper over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume either that $K_X + \Delta$ is \mathbb{Q} -Cartier and simple normal crossings over R , or is obtained after a finite number of steps of the minimal model program from such a \mathbb{Q} -divisor. Let V be the visible boundary of $\text{Nef}(X/R)$ from $v_0 \in N^1(X)_{\mathbb{R}}$ of the divisor $K_X + \Delta$. Then (1) every compact subset $F \subset \text{relint } V$ is contained in a union of finitely many supporting rational hyperplanes, and (2) every \mathbb{Q} -Cartier \mathbb{Q} -divisor on X with class in $\text{relint } V$ is semiample.

Proof. As in the cited theorem, noting it only requires the finite generation of Corollary 16, and Theorem 18. Part (2) is a consequence of Corollary 22. \square

9. BIG ABUNDANCE / \mathbb{R}

Let Γ a geometric valuation on X . Define

$$\sigma_\Gamma = \inf \{ \text{mult}_\Gamma D' \mid D \sim_{\mathbb{R}} D' \geq 0 \} \in \mathbb{R}.$$

As a result of Theorem 16, there is the following abundance theorem:

Corollary 22. [CL, 3.8] *Let (X, Δ) be a klt log smooth pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that Δ is big. If $K_X + \Delta$ is nef, then it is semi-ample.*

Proof. Let A an ample \mathbb{Q} -divisor on X such that $(X, \Delta + A)$ is klt. As Δ is big, it is possible to write $\Delta = \Delta' + A$ such that $(X, \Delta' + A)$ is klt. Thus the ring:

$$\mathcal{R} = R(K_X + \Delta, K_X + \Delta + A)$$

is finitely generated by Corollary 16, and $\text{Supp } \mathcal{R}$ contains a big divisor $(K_X + \Delta + A)$. Thus, any pseudo-effective

$$D \in \mathbb{R}_+(K_X + \Delta) + \mathbb{R}_+(K_X + \Delta + A)$$

is in $\text{Supp } \mathcal{R}$. By Theorem 18(2),

$$\text{Supp } \mathcal{R} = \mathbb{R}_+(K_X + \Delta, K_X + \Delta + A).$$

For any $\epsilon > 0$, and any geometric valuation Γ on X ,

$$\sigma_\Gamma(K_X + \Delta + \epsilon A) = 0$$

where, so that σ_Γ is zero on $\text{Supp } \mathcal{R}$, and thus the centre of Γ is not in $\mathbb{B}(K_X + \Delta)$ for any such Γ .

□

Corollary 23. *Let X be a normal projective variety of dimension 2 (over a perfect field k) and let D_1, \dots, D_r be \mathbb{Q} -Cartier \mathbb{Q} -divisors on X . Assume that the ring $R = R(X; D_1, \dots, D_r)$ is finitely generated, and let $\text{Supp } R = \bigcup C_i$ be a finite rational polyhedral subdivision such that for every geometric valuation Γ of X , σ_Γ is linear on C_i , as in Theorem 18. Denote $\varphi : \sum \mathbb{R}_+ D_i \rightarrow N^1(X)_\mathbb{R}$ the projection, and assume there exists k such that $C_k \cap \varphi^{-1}(\text{Amp } D) \neq \emptyset$. Then $C_k \subset \text{Supp } R \cap \varphi^{-1}(\text{Nef } X)$.*

Proof. As in the source, but applying Theorems 18 and 22. □

10. CONTRACTIONS GIVEN BY EXTREMAL RAYS / R

Lemma 24. [CL, 6.2] *Let (X, Δ) be a log smooth klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Suppose (X, Δ) is not nef, and A is a big \mathbb{Q} -divisor such that $(X, \Delta + A)$ is klt, and $K_X + \Delta + A$ is nef. Let λ be the nef threshold. Then $\lambda \in \mathbb{Q}^+$, and there is $R \subset \overline{NE}(X)$ with $(K_X + \Delta + \lambda A) \cdot R = 0$ and $(K_X + \Delta) \cdot R < 0$.*

Proof. As in the cited theorem, the only result being needed is the finite generation of Corollary 16. □

11. BIG TERMINATION WITH SCALING

Theorem 25. *Let (X, Δ) be a projective \mathbb{Q} -factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic*

$p > 0$ and perfect fraction field K . Suppose Δ is big and simple normal crossings over R . Let A be a sufficiently general ample \mathbb{Q} -divisor on X such that $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is nef. Then the $K_X + \Delta$ -minimal model program with scaling of A terminates.

Success is declared when there is a morphism $f : (X, \Delta) \rightarrow (X_n, \Delta_n)$ to a pair over a DVR such that on both geometric fibers $K_{X_n} + \Delta_n$ is nef. (Actually, applying [Liu, 5.3.24] and [Laz, 1.4.10], it suffices to show that the special fiber of the resulting model is nef).

Proof. This follows easily from [CL, 6.5], noting we have the required finite generation from Theorem 16. Let $\{\alpha_i\}$ be a sequence of numbers corresponding to a minimal model program defined on both fibers (α_i is the smallest positive real number such that $K_{X_i} + \Delta_i + \alpha_i A_i$ is nef on both fibers - although it suffices to check on the special fiber by Lemma 2). The claim is that the set of indices i is finite.

Choose r big divisors H_j arbitrarily close to $\Delta + \alpha$ for some norms on $N_1(X)$, and such that the $\text{span}\{K_X + H_j\}$ fills up all the dimensions of $N_1(X)$ and contains $K_X + \Delta + \alpha_1 A'$ on both fibers. Let C^1 be the \mathbb{R}_+ -cone spanned by the $K_{X_1} + H_i$. Let C^i denote the proper transform of C^1 , H_j^i the proper transforms of H_j , and let

$$\begin{aligned} R^i &= R(K_{X^i} + \Delta', K_{X^i} + H_1^i, \dots, K_{X^i} + H_r^i) \\ &\approx R^1 = R(K_X + \Delta, K_X + H_1, \dots, K_X + H_r) \end{aligned}$$

which are all finitely generated by Theorem 16 under the identification to R^1 .

By construction of the H^i , for each i , $G_i = K_{X_i} + \Delta'_i + \alpha_i A'_i \in \text{int } C^i$, and since G_i is nef, actually there are some ample divisors in $\text{int } C^i$. By Lemma 18, and since $\text{Supp } R^i = \bigcup C_\ell^i$, $\text{Supp } R^i = \bigcup C_\ell^i$ clearly contain big divisors, applying Theorem 19 (so that either G_i is \mathbb{Q} -Cartier so we can apply Theorem 18(3) or G_i is vacuously in $\text{Supp } R^i$) there are some ample divisors in C_ℓ^i for some ℓ . Thus using Theorem 23, the C_ℓ^i consist of the pullbacks of nef divisors from X_i . Since there are only finitely many cones in the subdivision $\text{Supp } R_1 = \bigcup C_\ell^1$, eventually there will exist an index i_0 , such that for $i > i_0$, the pullback of the nef cone of X_i lands in a C_ℓ^1 which is contained in the pullback of $\text{Nef}(X_{i'})$ for some $i' \leq i_0$. After this point, applying the negativity lemma (c.f. [Xu, 2.1] for positive characteristic and $\dim \leq 3$) as in [CL, 6.5] gives a morphism. \square

12. REDUCTIONS

Lemma 26. [HMX2014, 2.8.3] *Let (X, Δ) be a log smooth pair which is a scheme of dimension at most 3, with the coefficients of Δ belonging to $(0, 1]$, and with X projective. If (X, Δ) has a weak log canonical model, then there is a sequence $\pi : Y \rightarrow X$ of smooth blow ups of the strata of Δ such that if we write $K_Y + \Gamma = \pi^*(K_X + \Delta) + E$, where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$ and if we write*

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma(Y, K_Y + \Gamma),$$

then $\mathbb{B}_-(Y, K_Y + \Gamma')$ contains no strata of Γ' . If Δ is a \mathbb{Q} -divisor, then Γ' is a \mathbb{Q} -divisor.

Proof. Follows easily from the cited theorem. \square

Lemma 27. [GL, 2.3] *Let X/R a projective scheme of relative dimension 2 over a DVR R with algebraically closed residue field and perfect fraction field. Let (X, Δ) be a klt pair and $\varphi : W \rightarrow X$ a log resolution of (X, Δ) . Choose Δ_W so that $K_W + \Delta_W = \varphi^*(K_X + \Delta) + E$ with Δ_W and E effective \mathbb{Q} -Weil divisors with no common component. Let $F = \sum_{F_i \text{ } \varphi\text{-exceptional prime divisor}} F_i$ and $\Delta_W^\epsilon = \Delta_W + \epsilon F$, then (W, Δ_W^ϵ) is an ϵ -log smooth model of (X, Δ) . Then for $0 < \epsilon \ll 1$, a discrepancy negative (good) minimal model $\phi : (W, \Delta_W^\epsilon) \rightarrow (W_{\min}, \Delta_{W_{\min}}^\epsilon)$ with either ϕ a morphism or (X, Δ) log-smooth is also a (good) minimal model for (X, Δ) .*

Proof. Follows easily from [BCHM, 3.6.10, 3.6.11]. \square

Lemma 28. [HMX2014, 5.3] *Let X/R a projective scheme of relative dimension 2 over a DVR R , with algebraically closed residue field and perfect fraction field. Let (X, Δ) be a log smooth dlt pair with X \mathbb{Q} -factorial and projective and Δ a \mathbb{Q} -divisor. If Φ is a \mathbb{Q} -divisor such that*

$$0 \leq \Delta - \Phi \leq N_\sigma(X, K_X + \Delta),$$

then steps of the $K_X + \Phi$ -minimal model program are steps of the $K_X + \Delta$ minimal model program, and so termination for (X, Φ) implies termination for (X, Δ) .

Proof. (Similar proof to the cited Theorem). Let $f : X \rightarrow Y$ the result of running a $(K_X + \Phi)$ minimal model program and let $Y \rightarrow Z$ be the ample model of $K_X + \Phi$. Let $\Delta_t = t\Delta + (1 - t)\Phi$ so that if $0 < t \ll 0$, then f is also a $(K_X + \Delta_t)$ -MMP. If C generates a $K_X + \Delta_t$ -negative extremal ray, suppose that $(K_X + \Phi) \cdot C > 0$. Applying Theorem 9 on the special fiber,

$$-4 \leq (K_X + \Delta) \cdot C < 0.$$

Then, following the inequalities in [HMX2014, 5.1], gives a contradiction if $0 < t < \frac{1}{5}$. Thus every step of the $K_X + \Delta_t$ with scaling of an ample divisor is $(K_X + \Phi)$ -trivial, so after finitely many steps there is a model $g : X \rightarrow W$ contracting the components of $N_\sigma(X, K_X + \Delta_t)$ (this holds with no changes from [HMX2014, 2.7.1]). Thus, as $\text{supp}(\Delta - \Phi) \subset \text{supp} N_\sigma(X, K_X + \Delta) = \text{supp} N_\sigma(X, K_X + \Delta_t)$, then $g_*(K_Y + \Phi)$ is semiample whenever $g_*(K_X + \Delta)$ is, and [HMX2014, 2.7.2] (which holds using only the log smooth resolution) implies that g is a minimal model of (X, Δ) , which is good when (X, Φ) has a good minimal model.

□

13. EXISTENCE OF MINIMAL MODELS SPECIAL CASE

Lemma 29. [HMX2010, 3.1] *Let (X, Δ) be a \mathbb{Q} -factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k) . Let $f : X \rightarrow Y$ be a step of the $(K_X + \Delta)$ -MMP. If f is birational and V is a non-canonical*

centre of (X, Δ) , then V is not contained in the indeterminacy locus of f , V_0 is not contained in the indeterminacy locus of f_0 , and the induced maps $\phi : V \rightarrow W$ and $\phi_k : V_k \rightarrow W_k$ are birational, where $W = f(V)$. Let $\Gamma = f_*\Delta$. Then $\mathbb{B}_-(Y_0, K_{Y_0} + \Gamma_0)$ contains no non-canonical centres of (Y_0, Γ_0) (so we may repeat the process), and if V is a non-klt centre or $V = X$, then $\phi : V \rightarrow W$ and $\phi_0 : V_0 \rightarrow W_0$ are birational contractions. If f is a Mori fibre space, then f_0 is not birational.

Proof. (Follows easily from the cited Theorem). Suppose f is birational. Let V be a non-canonical centre of (X, Δ) . Let $g : X \rightarrow Z$ be the contraction of the extremal ray associated to f (so that $f = g$ unless f is a flip). Let $Q = g(V)$, and let $\psi : V \rightarrow Q$, be the induced morphism. As every component of V_k is a non-canonical centre of (X_k, Δ_k) , then by hypothesis, components of V_k are not contained in $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$, thus ψ_k is defined at each such component, hence is birational. By upper-semicontinuity of the fibers of ψ , ψ is birational, and thus $\phi : V \rightarrow Q \rightarrow W$, and ϕ_k are both birational.

Now suppose V is a non-klt centre or $V = X$, the above holds in the first case, as non-klt centres are non-canonical. Comparing the discrepancies of the differentials of adjunction for Δ_k and Γ_k as in the cited proof shows that ϕ_k, f_k , and thus ϕ are birational contractions. On the other hand, if f is a Mori fibre space, then, as the dimension of the fibers of $f : X \rightarrow Y$ are upper-semicontinuous, f_k is not birational. \square

Theorem 30. [HMX2014, 3.2] *Let (X, Δ) be a \mathbb{Q} -factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k*

of characteristic $p > 0$ and perfect fraction field K . Assume that $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k) and is log smooth. Let $f : X \rightarrow Y$ be a step of the $(K_X + \Delta)$ -MMP. Then the minimal model program with scaling terminates using only contractions.

Proof. (Similar proof to the cited theorem). Let $f : X \rightarrow Y$ the $(K_X + \Delta)$ -MMP with scaling of an ample divisor A . Let $\Gamma = f_*\Delta$ and $B = f_*A$. By construction, $K_Y + tB + \Gamma$ is nef for some $t > 0$. By Lemma 29, $f : X \rightarrow Y$ is a birational contraction and $f_k : X_k \rightarrow Y_k$ is a birational contractions from $(X_k, tA_k + \Delta_k)$. If $K_X + \Delta$ is not pseudo-effective, then for $t > 0$, the result of f is a Mori fiber space, and by Lemma 29, Y_k is covered by $K_{Y_k} + tB_k + \Gamma_k$ -negative curves, contradicting bigness of $K_{X_k} + tA_k + \Delta_k$. Thus $K_X + \Delta$ is pseudo-effective, and given any $\epsilon > 0$, we may run the MMP until $t < \epsilon$. Now we conclude by Theorem 17. Letting ϵ be the constant given in that theorem, the minimal models for $(X_k, tA_k + \Delta_k)$ are all isomorphic $t \in [0, \epsilon]$. Thus, once $t < \epsilon$, any more steps in the minimal model program with scaling must be an isomorphism on the special fiber, and thus an isomorphism. Thus there exists a minimal model (Y, Γ) for (X, Δ) . \square

14. EXISTENCE OF MINIMAL MODELS GENERAL CASE

Theorem 31. *Let (X, Δ) be a klt pair of relative dimension 2, proper over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $K_X + \Delta$ is \mathbb{Q} -Cartier, and log smooth over R . Then the minimal model of (X, Δ) exists.*

Proof. I start by repeating the reduction of [HMX2014, 6.1]. If necessary, extend R to be complete with algebraically closed residue field k , so the strata of Δ have irreducible fibers over R . Let $f_0 : Y_0 \rightarrow X_0$ be the birational morphism of Lemma 26. Under the log smooth hypothesis, and since the strata of Δ have irreducible fibers, and f_0 blows up strata of Δ_0 , extend f_0 to a birational morphism $f : Y \rightarrow X/R$ which is a composition of smooth blow ups of strata of Δ . Write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E$$

with $\Gamma \geq 0$ and $E \geq 0$, $f_*\Gamma = \Delta$, and $f_*E = 0$. Then (Y, Γ) is log smooth and the fibres of components of Γ are irreducible. By Lemma 27, (X, Δ) has a minimal model if (Y, Γ_ϵ) has a good minimal model, where $(Y, \Gamma_\epsilon) = (Y, \Gamma + \epsilon F)$ is the ϵ -log smooth model of (X, Δ) with F the sum of f -exceptional divisors and $0 < \epsilon \ll 1$.

Replace (X, Δ) by (Y, Γ_ϵ) and set $\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma(X_k, K_{X_k} + \Delta_k)$ so that $\mathbb{B}_-(X_k, K_{X_k} + \Theta_k)$ contains no strata of Θ_k . Let $0 \leq \Theta \leq \Delta$ be the unique divisor such that $\Theta_k = \Theta|_{X_k}$. Let H relatively ample such that $(X, \Theta + H)$ is log smooth over R , and $K_X + \Theta + H$ is big. There is a commutative diagram:

$$\begin{array}{ccc} \pi_*\mathcal{O}_X(m(K_X + \Theta) + H) & \longrightarrow & \pi_*\mathcal{O}_X(m(K_X + \Delta) + H) \\ \downarrow & & \downarrow \\ H^0(X_k, \mathcal{O}_{X_k}(m(K_{X_k} + \Theta_k) + H_k)) & \longrightarrow & H^0(X_k, \mathcal{O}_{X_k}(m(K_{X_k} + \Delta_k) + H_k)) \end{array}$$

with surjective columns by Theorem 14 and [Liu, 5.3.20(b)], and with the bottom row an isomorphism. Applying Nakayama's lemma gives

an isomorphism on the top row, so that $\Theta \geq \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)$. Again applying Theorem 14, gives $\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)$. Thus $\Delta - \Theta \leq N_\sigma(X, K_X + \Delta)$, so by Lemma 28, it suffices to find a minimal model for (X, Θ) . Replacing (X, Δ) by (X, Θ) , it suffices to assume that $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no strata of Δ_0 . Letting A be an ample divisor, we run the minimal model program with scaling of A . Now the assumptions of Theorem 30 apply, so we know that (X, Θ) has a minimal model.

□

REFERENCES

- [A1] Artin, M. "Algebraization of formal moduli. I, Global Analysis (Papers in Honor of K. Kodaira), 21–71." (1969).
- [A2] Artin, Michael. "Algebraization of formal moduli: II. Existence of modifications." *Annals of Mathematics* (1970): 88-135.
- [BC] Baker, Matthew., and Janos A. Csirik. "On the Isomorphism Between the Dualizing Sheaf and the Canonical Sheaf." (1996).
- [BCHM] Birkar, Caucher, et al. "Existence of minimal models for varieties of log general type." *Journal of the American Mathematical Society* 23.2 (2009): 405.
- [Bir] Birkar, Caucher. "Existence of flips and minimal models for 3-folds in char p." arXiv preprint arXiv:1311.3098 (2013).
- [Bo] Bosch, Siegfried, et al. "Néron models." (1990).
- [BP] Berndtsson, Bo, and Mihai Paun. "Quantitative extensions of pluricanonical forms and closed positive currents." *Nagoya Math. J* 205 (2012): 25-65.
- [CL] Corti, Alessio, and Vladimir Lazić. "New outlook on the minimal model program, II." *Mathematische Annalen* 356.2 (2013): 617-633.

- [CP2009] Cossart, Vincent, and Olivier Piltant. "Resolution of singularities of threefolds in positive characteristic II." *Journal of Algebra* 321.7 (2009): 1836-1976.
- [CP2014] Cossart, Vincent, and Olivier Piltant. "Resolution of Singularities of Arithmetical Threefolds II." arXiv preprint arXiv:1412.0868 (2014).
- [CR] Chatzistamatiou, Andre, and Kay Rülling. "Higher direct images of the structure sheaf in positive characteristic." *Algebra & Number Theory* 5.6 (2012): 693-775.
- [CZ] Cascini, Paolo, and DEQI ZHANG. "Effective finite generation for adjoint rings." arXiv preprint arXiv:1203.5204 (2012).
- [Das] Das, Omprokash. "On Strongly F -Regular Inversion of Adjunction." arXiv preprint arXiv:1310.8252 (2013).
- [Deb] Dèbes, Pierre, et al., eds. *Arithmetic and geometry around Galois theory*. Springer Science & Business Media, 2012.
- [DF] Di Cerbo, Gabriele, and Andrea Fanelli. "Effective Matsusaka's Theorem for surfaces in characteristic p ." arXiv preprint arXiv:1501.07299 (2015).
- [DI] Deligne, Pierre, and Luc Illusie. "Relèvements modulop 2 et décomposition du complexe de de Rham." *Inventiones Mathematicae* 89.2 (1987): 247-270.
- [EGSurfDIOP] Egbert, Andrew. SurfDIOP. <https://divisibility.wordpress.com/2015/06/12/mixed-characteristic-minimal-models-and-invariance-of-plurigeners-for-relative-dimension-2/>
- [EgSyz] Egbert, Andrew. <https://divisibility.wordpress.com/2015/08/07/adjoint-rings-and-szygies-in-mixed-or-positive-characteristic/>
- [ELM] Ein, Lawrence, et al. "Asymptotic invariants of base loci." *Annales de l'institut Fourier*. Vol. 56. No. 6. 2006.

- [Ek] Ekedahl, Torsten. "Canonical models of surfaces of general type in positive characteristic." *Publications Mathématiques de l'IHÉS* 67.1 (1988): 97-144.
- [EV] Viehweg, Eckart. *Lectures on vanishing theorems*. Vol. 20. Springer Science & Business Media, 1992. (The online lecture notes version).
- [GH] Griffiths, Phillip, and Joseph Harris. *Principles of algebraic geometry*. John Wiley & Sons, 2014.
- [GL] Gongyo, Yoshinori, and Brian Lehmann. "Reduction maps and minimal model theory." *Compositio Mathematica* 149.02 (2013): 295-308.
- [Gra] Grauert, Hans, Thomas Peternell, and Reinhold Remmert. *Several complex variables VII: sheaf-theoretical methods in complex analysis*. Vol. 7. Springer Science & Business Media, 1994.
- [Gr] Grothendieck, A. "Éléments de géométrie algébrique." New York (1967).
- [GW] Görtz, Ulrich, and Torsten Wedhorn. *Algebraic Geometry*. Vieweg+Teubner, 2010.
- [HK] Hacon, Christopher D., and Sándor Kovács. *Classification of higher dimensional algebraic varieties*. Vol. 41. Springer Science & Business Media, 2011.
- [Hara98] Hara, Nobuo. "Classification of two-dimensional F-regular and F-pure singularities." *Advances in Mathematics* 133.1 (1998): 33-53.
- [Har] Hartshorne, Robin. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 1977.
- [HS] Hindry, Marc, and Joseph H. Silverman. *Diophantine geometry: an introduction*. Vol. 201. Springer Science & Business Media, 2000.
- [HMX2010] Hacon, Christopher, James McKernan, and Chenyang Xu. "On the birational automorphisms of varieties of general type." arXiv preprint arXiv:1011.1464 (2010).

- [HMX2014] Hacon, CHRISTOPHER D., James McKernan, and Chenyang Xu. "Boundedness of moduli of varieties of general type." arXiv preprint arXiv:1412.1186 (2014).
- [HX] Hacon, Christopher D., and Chenyang Xu. "Existence of log canonical closures." *Inventiones mathematicae* 192.1 (2013): 161-195.
- [KU] Katsura, Toshiyuki, and Kenji Ueno. "On elliptic surfaces in characteristic." *Mathematische Annalen* 272.3 (1985): 291-330.
- [KK94] Kollár, János, and Sándor Kovács. "Birational geometry of log surfaces." preprint (1994).
- [KM] Kollár, János, and Shigefumi Mori. *Birational geometry of algebraic varieties*. Vol. 134. Cambridge University Press, 2008.
- [Ko] Kollár, János. *Singularities of the minimal model program*. Vol. 200. Cambridge University Press, 2013.
- [Ko2] Kollár, János. *Shafarevich maps and automorphic forms*. Princeton University Press, 2014.
- [Ko1991] Kollár, János, ed. *Flips and abundance for algebraic threefolds: a summer seminar at the University of Utah, Salt Lake City, 1991*. Société mathématique de France, 1992.
- [Lai] Lai, Ching-Jui. "Varieties fibered by good minimal models." *Mathematische Annalen* 350.3 (2011): 533-547.
- [Laz] Lazarsfeld, Robert K. *Positivity in algebraic geometry*. Springer Science & Business Media, 2004.
- [Lev] Levine, Marc. "Pluri-canonical divisors on Kähler manifolds." *Inventiones mathematicae* 74.2 (1983): 293-303.
- [Lev-2] Levine, Marc. "Pluri-canonical divisors on Kähler manifolds, II." *Duke Math. J* 52.1 (1985): 61-65.
- [Lie] Liedtke, Christian. *Algebraic surfaces in positive characteristic*. Birational geometry, rational curves, and arithmetic. Springer New York, 2013. 229-292.

- [Liu] Liu, Qing, and Reinie Erne. Algebraic geometry and arithmetic curves. Oxford university press, 2002.
- [LS] Liedtke, Christian, and Matthew Satriano. "On the birational nature of lifting." *Advances in Mathematics* 254 (2014): 118-137.
- [Ma] Maddock, Zachary. "A bound on embedding dimensions of geometric generic fibers." arXiv preprint arXiv:1407.2529 (2014).
- [MP] Manin, Yuri Ivanovic, and Alexei A. Panchishkin. Introduction to modern number theory. Vol. 49. Springer-Verlag Berlin Heidelberg, 2005.
- [Mck] Mckernan, James. Course Lecture Notes. http://math.mit.edu/~mckernan/Teaching/07-08/Autumn/18.735/1_7.pdf
- [Mum] Mumford, David, Chidambaram Padmanabhan Ramanujam, and Juri Ivanovič Manin. Abelian varieties. Vol. 48. Oxford: Oxford university press, 1970.
- [Oss] Osserman, Brian. "Notes on Cohomology and Base Change." <https://www.math.ucdavis.edu/~osserman/math/cohom-base-change.pdf>
- [Sat] Satriano, Matthew. "De Rham theory for tame stacks and schemes with linearly reductive singularities." arXiv preprint arXiv:0911.2056 (2009).
- [Serre] Serre, Jean-Pierre. Local fields. Vol. 67. Springer Science & Business Media, 2013.
- [Siu] Siu, Yum-Tong. "Invariance of plurigenera." arXiv preprint alg-geom/9712016 (1997).
- [Siu2] Siu, Yum-Tong. "Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type." *Complex geometry*. Springer Berlin Heidelberg, 2002. 223-277.

- [Suh] Suh, Junecue. "Plurigenera of general type surfaces in mixed characteristic." *Compositio Mathematica* 144.05 (2008): 1214-1226.
- [Tan] Tanaka, Hiromu. "Minimal models and abundance for positive characteristic log surfaces." *Nagoya Mathematical Journal* (2015).
- [Tan2] Tanaka, Hiromu. "The X-method for klt surfaces in positive characteristic." *arXiv preprint arXiv:1202.2497* (2012).
- [Tan3] Tanaka, Hiromu. "The trace map of Frobenius and extending sections for threefolds." *arXiv preprint arXiv:1302.3134* (2013).
- [Tan4] Tanaka, Hiromu. "Minimal model theory for surfaces over an imperfect field." *arXiv preprint arXiv:1502.01383* (2015).
- [Ter] Terakawa, Hiroyuki. "The d -very ampleness on a projective surface in positive characteristic." *Pacific J. of Math* 187 (1999): 187-198.
- [Wal] Waldron, Joe. "Finite generation of the log canonical ring for 3-folds in char p ." *arXiv preprint arXiv:1503.03831* (2015).
- [Xu] Xu, Chenyang. "On base point free theorem of threefolds in positive characteristic." *arXiv preprint arXiv:1311.3819* (2013).