

LOG MINIMAL MODELS FOR ARITHMETIC THREEFOLD GERMS

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ABSTRACT. In this note, I attempt to finish the log minimal model program proofs started in [EGSurfDIOP] and [EgSyz]. In [EGSurfDIOP], finite generation of the canonical ring is proven for klt, log smooth pairs of general type on an algebraic space having two dimensional fibers over a DVR. In the [EgSyz], the minimal model program for such pairs is studied. In this note, I relax the assumption on kodaira dimension. Most of this paper is just using results from characteristic 0, checking what changes are needed for them to hold over a mixed characteristic DVR, and then applying the finite generation in the big case (from [EGSurfDIOP]).

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1. INTERSECTION THEORY ON X/R

Suppose X/R is a proper algebraic space of relative dimension 2 over a discrete valuation ring R with residue field k of characteristic $p > 0$ and perfect fraction field K (See for example the compact Shimura Varieties mentioned in [Suh, 0.1].)

Lemma 1. [KU, 9.3] *Let $\varphi : X \rightarrow \text{Spec}(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $\text{spec}(R)$, where R is a DVR with algebraically closed residue field k , and field of fractions K . Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ . If D, D' are divisors on X , then*

$$\begin{aligned} K_{X_k}^2 &= K_{X_K}^2 \\ (K_{X_k} \cdot D_k)_{X_k} &= (K_{X_K} \cdot D_K)_{X_K} \\ (D_k \cdot D'_k)_{X_k} &= (D_K \cdot D'_K)_{X_K}. \end{aligned}$$

If X/R is smooth, then Lemma 1 applies to show that for any divisors C, D extending to both fibers, we can define $C \cdot D = (C|_{X_k} \cdot D|_{X_k})_{X_k}$. On the other hand, if X/R is merely normal and proper, but is actually a scheme, the resolution of singularities, Theorem 3, holds. Thus the

intersection theory can be defined as in [Tan, Def 3.1]: $f : X' \rightarrow X$ is a resolution, and $C.D = f^*C.f^*D$ for two divisors C, D on Y/R . By properness, this intersection extends by linearity to Weil divisors with \mathbb{Q} or \mathbb{R} coefficients. Numerical equivalence and $N^1(X)_{\mathbb{Q}, \mathbb{R}}$ are then defined as usual.

Note the two different hypothesis here: The invariance of plurigenera theorem requires a proper, smooth algebraic space of relative dimension 2, while the existence of minimal models requires the intermediate steps of the minimal model program to have some intersection theory, and thus be schemes. Recall that a Cartier divisor C on a variety is usually called nef if $C.D \geq 0$ for all curves C .

Lemma 2. *Let S be an affine Dedekind scheme and $f : X \rightarrow S$ a projective morphism. Let \mathcal{L} be an invertible sheaf on X such that \mathcal{L}_s is nef for every closed point $s \in S$. Then \mathcal{L} is nef.*

Proof. By [Liu, 5.3.24], the statement holds when nef is replaced by ample. Now, c.f. [Laz, 1.4.10], it suffices to choose an ample divisor H which restricts to X_s , so that $D_s + \epsilon H_s$ is ample for all sufficiently small ϵ . Then $D + \epsilon H$ is ample for all sufficiently small ϵ , and so D is nef. \square

2. RESOLUTION

Theorem 3. [CP2014, 1.1] *Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \rightarrow X$ with the following properties:*

- (i) X' is everywhere regular
- (ii) π induces an isomorphism $\pi^{-1}(\text{Reg } X) \approx \text{Reg } X$
- (iii) $\pi^{-1}(\text{Sing } X)$ is a strict normal crossings divisor on X' .

3. EFFECTIVE KLT KAWAMATA VIEHWEG VANISHING

In this section, I note that a simple modification of Tanaka's vanishing theorem [Tan2] gives a version of the (log) Kawamata Viehweg Vanishing in which the multiple of the nef divisor is at least computable. There is a similar recent result in [DF], which is more restrictive in that it requires a smooth variety rather than a normal projective klt pair. First, a Theorem of Terakawa:

Theorem 4. [Ter] *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X . Assume that either*

- (1) $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or
- (2) X is of general type with
 $p \geq 3$ and $(D^2) > \text{vol}(X)$ or
 $p = 2$ and $(D^2) > \max\{\text{vol}(X), \text{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all $i > 0$.

Applying the above to Tanaka's proof gives the following:

Theorem 5. *Effective KLT Kawamata-Viehweg Vanishing for normal surfaces) Let (X, Δ) be a normal projective klt surface over an*

algebraically closed field of characteristic $p > 2$, where Δ is an effective \mathbb{R} -divisor. Let N be a nef and big \mathbb{R} -cartier \mathbb{R} -divisor with $N^2 > \max\{\text{vol}(X), \text{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$. Let D be a \mathbb{Q} -Cartier \mathbb{Z} -divisor such that $D - (K_X - \Delta)$ is nef and big. Then

$$H^i(X, \mathcal{O}_X(D + rN + N')) = 0$$

for every $i > 0$, every positive real number $r \geq 1$, and every nef \mathbb{R} -Cartier \mathbb{R} -divisor N' such that $rN + N'$ is a Cartier divisor.

Proof. Verbatim from [Tan2, 2.11] except that Weak Kodaira Vanishing [Tan2, 2.4] in the proof of [Tan2, 2.6] is replaced by Terakawa's vanishing theorem 4.

□

4. INVARIANCE OF PLURIGENERA SPECIAL CASE

Lemma 6. [KU, 9.4] *Let $\varphi : X \rightarrow \text{Spec}(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $\text{spec}(R)$, where R is a DVR with algebraically closed residue field k , and field of fractions K . Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ . If X_k contains an exceptional curve of the first kind e , there exists a DVR $\tilde{R} \supset R$, with residue field isomorphic to k , and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \rightarrow \text{Spec}(\tilde{R})$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \rightarrow \tilde{Y}$ over $\text{Spec}(\tilde{R})$ such that on the closed fibre, π induces the contraction of the exceptional curve e . Moreover, on the generic*

fibres, π also induces a contraction of an exceptional curve of the first kind.

Proof. (Follows easily from the cited theorem, although I make a small observation about the extension of residue fields). By [A1, Cor 6.2] $\text{Hilb}_{X/\text{Spec}(R)}$ is represented by an algebraic space \mathcal{H} which is locally of finite type over $\text{Spec}(R)$. Let Y be the irreducible component containing the point $\{e\}$ corresponding to the exceptional curve e on the special fiber. Then $e \approx \mathbb{P}_k^1$ and $N_{e/X} \approx \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, so Y is regular at $\{e\}$ and of dimension 1. Since e is fixed in the special fiber, the structure morphism $p : Y \rightarrow \text{Spec}(R)$ is surjective. By [Bo, 2.2.14]¹, we can find an étale cover $\tilde{R} \supset R$ and a morphism $j : \text{Spec}(\tilde{R}) \rightarrow Y$ over $\text{Spec}(R)$ with $j(\tilde{e}) = \{e\}$ (if R is not already complete, then first extend R to a complete DVR using [Gr, 0.6.8.2,3] so that \tilde{R} is again a DVR). As k is assumed algebraically closed, and $\tilde{R} \rightarrow R$ is unramified, then the extension of residue fields is finite and separable at the closed point of R , and hence an isomorphism of residue fields. Let $\hat{p} : \mathcal{E} \rightarrow \text{Spec}(\tilde{R})$ be the pull-back of the universal family over Y . As the closed fibre \mathcal{E}_0 is a projective line, we may choose the morphism j in such a way that the generic fibre \mathcal{E}_1 of \hat{p} is also a projective line. Moreover, \mathcal{E} can be considered as a smooth closed algebraic subspace of codimension 1 in $\hat{X} = X \otimes \tilde{R}$. By Lemma 1, $-1 = \mathcal{E}_0^2 = \mathcal{E}_1^2$. Hence \mathcal{E}_1 is also an exceptional curve of the first kind. Hence by [A2, Cor 6.11] there exists a contraction morphism $\pi : \hat{X} \rightarrow \hat{X}$ over $\text{Spec}(\tilde{R})$

¹If k is perfect, the proof of the cited lemma seems to go through when $f : X \rightarrow S$ is merely smooth at $x \in X$ and $f : X \rightarrow S$ is locally of finite type.

which contracts \mathcal{E} to a section of $\tilde{\varphi} : \tilde{X} \rightarrow \text{Spec}(R)$ where $\tilde{\varphi}$ is proper, smooth, separated and of finite type over $\text{Spec}(\tilde{R})$.

□

Lemma 7. [KK94, 2.3.5] *Let (S, B) be a log-canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C \cdot (K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

Theorem 8. [Tan, 4.4, 4.6] *Let X be a projective normal surface and let Δ be an effective \mathbb{R} -divisor such that $K_X + \Delta$ is \mathbb{R} -cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let H be an \mathbb{R} -cartier ample \mathbb{R} -divisor. Then the following assertions hold:*

- (1) $\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0} [C_i]$
- (2) $\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta + H \geq 0} + \sum_{finite} \mathbb{R}_{\geq 0} [C_i]$
- (3) Each C_i in (1) and (2) is rational or $C_i = B_j$ for some B_j with $B_j^2 < 0$.
- (4) Each C_i in (1) and (2) satisfies $0 < -C_i \cdot (K_X + \Delta) \leq 3$.

Lemma 9. [Tan, 5.3] *Let X be a projective normal surface and let C be a curve in X such that $r(K_X + C)$ is Cartier for some positive integer r .*

- (1) If $C \cdot (K_X + C) < 0$, then $C \approx \mathbb{P}^1$.
- (2) If $C \cdot (K_X + C) = 0$, then $C \approx \mathbb{P}^1$ or $\mathcal{O}_C \left((K_X + C)^{[r]} \right) \approx \mathcal{O}_C$.

Proposition 10. *Let (X, Δ) be a terminal log smooth pair of relative dimension 2 over a DVR R with algebraically closed residue field k of characteristic $p > 2$ and perfect fraction field K . Assume that $K_X + \Delta$ is \mathbb{Q} -Cartier and $\nu(K_X + \Delta) \neq 1$.² Assume that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$. Then there exists a computable m_0 such that for $m > m_0$,*

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

The proof of proposition 10 is given in the following claims.

Claim. Assumptions as above, after passing to an extension R' of R , there is a proper, smooth algebraic space X^{min}/R' and an R' morphism $X \otimes R' \rightarrow X^{min}$ such that both $X_k \rightarrow X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef.

Proof. As k is algebraically closed of characteristic $p > 0$, then by the Cone Theorem 8,

$$\overline{NE}(X_k) = \overline{NE}(X_k)_{K_{X_k} + \Delta_k \geq 0} + \sum \mathbb{R}_{\geq 0} [C_i]$$

with each C_i is rational or $C_i = B_j$ for some B_j a component of Δ with $B_j^2 < 0$. Under the assumption that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$, then actually each C_i is rational and is not a component of Δ_k . Thus $C_i \cdot \Delta_k \geq 0$ so $C_i \cdot K_{X_k} < 0$, and since $N_\sigma(K_{X_k} + \Delta_k)$ is a \mathbb{Q} -divisor by [Tan, 7.1], then Theorem 9 implies $C_i \approx \mathbb{P}^1$. By Lemma 7, C_i can be contracted to a log-canonical point, which is actually a smooth point under the terminal assumption. Thus C_i is an exceptional curve of the first kind, so it is possible to apply Lemma 6.

²Only case that $\kappa(K_X + \Delta) = 2$ is needed for the rest of this paper.

By Lemma 6, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \rightarrow \tilde{X}$ induces the contraction of C_i on X_k and X_K , and further $\tilde{\varphi} : \tilde{X} \rightarrow \text{Spec}(\tilde{R})$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since k is algebraically closed, it induces identity on residue fields. Now I need to work with \tilde{X} . In order to apply the Cone Theorem 8 again, I need \tilde{X}_k to be projective, but a smooth algebraic space of dimension 2, proper, separated, and of finite type over an algebraically closed field is projective, so \tilde{X}_k is projective. Thus the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no problem in extending R . As the Picard number drops each step, there are only finitely many steps. \square

Claim 11. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef.

Proof. It suffices to apply Lemma 2 to X_k^{min} , and then restrict to X_K . \square

Now we proceed by cases depending on the Kodaira dimension.

Claim 12. In the case that (X_k, Δ_k) is big or has $\nu = 0$, there is a computable m_0 such that for $m_0 | m$, we have

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

Proof. Suppose first that (X_k, Δ_k) is big. By the above, we achieve X^{min} such that $K_X + \Delta$ is nef on both X_k^{min} and X_K^{min} . By the Kawamata Viehweg Vanishing Theorem 5 applied to the special fiber (and the semicontinuity theorem) there exists a computable $m_0 \gg 0$ such that for $m_0 < m$ and $i > 0$, we have

$$h^i(m(K_{X_K} + \Delta_K)) = h^i(m(K_{X_k} + \Delta_k)) = 0.$$

In fact, if $\Delta = 0$, then Ekedahl's vanishing Theorem [Ek] can be applied. Thus, applying the invariance of Euler characteristic [Liu, 5.3.22], and birational invariance of the plurigenera, it follows that for $m > m_0$,

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

Now suppose that $\nu(K_{X_k} + \Delta_k) = 0$. As before, we achieve an $\pi' : X^{min} \rightarrow R'$ and an R' morphism $X \otimes R' \rightarrow X^{min}$ such that both $X_k \rightarrow X_k^{min}$ is obtained by successive blow-downs of (-1) curves and both $K_{X_k^{min}} + \Delta_{X_k^{min}}$ and $K_{X_K^{min}} + \Delta_{X_K^{min}}$ are nef.

Applying [Tan], we find that $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is semi-ample. Under the pseudo-effective assumption, and since $\nu(K_{X_k^{min}} + \Delta_{X_k^{min}}) = 0$, but $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef, then actually $K_{X_k^{min}} + \Delta_{X_k^{min}} \equiv 0$. A high multiple of a semi-ample and numerically trivial bundle is globally generated and numerically trivial, hence $\mathcal{O}_{X_k^{min}}(m(K_{X_k^{min}} + \Delta_{X_k^{min}})) = \mathcal{O}_{X_k^{min}}$, $m_k \gg 0$. Actually the same holds on both fibers, since the abundance has been proven for a two-dimensional variety. Thus, perhaps taking $m' = m_k \cdot m_K$, then the conclusion follows from the cohomological flatness of the smooth map $X^{min} \rightarrow R'$, c.f. [Liu, Exc. 5.3.14].

□

5. INVARIANCE OF PLURIGENERA AND FINITE GENERATION

Theorem 13. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $K_X + \Delta$ is big, \mathbb{Q} -Cartier, and simple normal crossings over R . Then, there exists a (computable) m_0 depending on the intersection numbers such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

Thus, after possibly extending R , the canonical ring $R(K_X + \Delta)$ is finitely generated over R . If $K_X + \Delta$ is pseudo-effective (not necessarily big), a contraction of a $K_{X_k} + \Delta_k$ -negative extremal ray cut out by an ample H over R induces a contraction of a $K_X + \Delta$ negative extremal ray. If the contraction is a flipping contraction, then the flip exists.

Proof. Since the hypothesis and conclusion are preserved by base change, then after extending R if necessary, we may assume that k is algebraically closed and that R is complete. The proof follows similarly to [HMX2010, 1.6] except that Proposition 10 is used in place of Kawamata Viehweg Vanishing. Recall that the log-canonical ring of $K_{X_k} + \Delta_k$ is finitely generated (c.f. [Tan, 7.1]) so $N_\sigma(K_{X_k} + \Delta_k)$ is a \mathbb{Q} -divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma(K_{X_k} + \Delta_k)$$

and let $0 \leq \Theta \leq \Delta$ be a \mathbb{Q} -divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. Replacing (X, Δ) by a blow-up, assume (X, Δ) is terminal.

Letting $m \gg 0$ to fit the hypothesis of Proposition 10, and sufficiently divisible such that $m(K_{X_k} + \Theta_k)$ is integral, then by definition of N_σ ,

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_k} + \Theta_k)).$$

Furthermore, by Proposition 10,

$$h^0(m(K_{X_k} + \Theta_k)) = h^0(m(K_{X_K} + \Theta_K)).$$

As $\Theta \leq \Delta$, the first part of the theorem follows by semicontinuity.

Now by [Tan], there is $m \gg 0$, such that

$$\bigoplus_{m \geq 0} H^0(ml(K_{X_k} + \Delta_k))$$

is generated in degree 1, so that there is a surjection

$$S^l H^0(m(K_{X_k} + \Delta_k)) \twoheadrightarrow H^0(ml(K_{X_k} + \Delta_k)).$$

Consider the following diagram:

$$\begin{array}{ccc} S^l H^0(m(K_X + \Delta)/R) & \xrightarrow{\alpha} & H^0(ml(K_X + \Delta)/R) \\ \downarrow & & \downarrow \\ S^l H^0(m(K_{X_k} + \Delta_k)) & \twoheadrightarrow & H^0(ml(K_{X_k} + \Delta_k)) \end{array}$$

The vertical maps are surjective by the first part of the theorem (and applying Nakayama's lemma to pull back the generators of the k -modules). Now applying the version of Nakayama's Lemma given in [GH, Chapter 5.3], surjectivity of the bottom map implies surjectivity of α , and thus $R(K_X + \Delta)$ is finitely generated over R .

Finally, suppose that there is a contraction of a curve on the closed fiber (X_k, Δ_k) corresponding to a $K_{X_k} + \Delta_k$ -negative extremal ray Σ_k . Let $H_k = H|_{X_k}$ be such that $(K_{X_k} + \Delta_k + H_k) \cdot \Sigma_k = 0$. The goal is to show that

$$R(K_X + \Delta + H/R)$$

is finitely generated so that there exists a contraction defined by $X_1 = \text{Proj } R(K_X + \Delta + H/R)$. However, as H is ample, then under the assumption that $K_{X_k} + \Delta_k$ is pseudo-effective, $K_{X_k} + \Delta_k + H_k$ is actually big, and so this follows. \square

6. FINITE GENERATION (BIG CASE)

Corollary 14. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $K_X + \Delta$ is \mathbb{Q} -Cartier, and simple normal crossings over R . If A_1, \dots, A_m are any ample divisors, then the ring $R(X, K_X + \Delta + A_1, \dots, K_X + \Delta + A_n)$ is finitely generated over R .*

Proof. C.f. the proof of [CZ, 2.11], it suffices to prove that if $G = \sum_{i=1}^k m_i (a(K_X + \Delta_i + A_i))$, with fixed $a \in \mathbb{N}$, $m_1, \dots, m_k \geq 0$, and $l \in \{1, \dots, l\}$ then there exists m_l such that the following maps are surjective

$$\begin{aligned} & H^0(X, \mathcal{O}_X(G - a(K_X + \Delta - A_l))) \otimes H^0(X, \mathcal{O}_X(a(K_X + \Delta + A_l))) \\ & \rightarrow H^0(X, \mathcal{O}_X(G)). \end{aligned}$$

By Nakayama's Lemma applied as above, it suffices to show the surjectivity on the central fiber.

Applying [Tan], there exists a minimal model (X'_k, Δ'_k) for (X_k, Δ_k) such that $K_{X'_k} + \Delta'_k$ is nef and semi-ample. Thus there is $m \gg 0$ such that $m(K_{X'_k} + \Delta'_k)$ is both base-point free, and $m - 2$ is large enough so the effective Kawamata Viehweg Vanishing Theorem 5 applies. Now the proof of [CZ, 2.7] applies. \square

7. FINITENESS OF MODELS FOR SURFACES

Theorem 15. *Let X be a two dimensional normal variety over a perfect field of characteristic $p > 0$. Let (X, Δ) be a log smooth KLT pair and A an ample \mathbb{Q} -divisor on X . Then there exists an $\epsilon > 0$ such that the minimal models (and the output of the minimal model program with scaling) for $(X, \Delta + tA)$ are all isomorphic for $t \in [0, \epsilon]$.*

Proof. (Essentially same as [Lai, 26,27] and simplifies due to the fact that a birational isomorphism in codimension 1 on a surface gives an isomorphism. There is also a proof of essentially the same result in [HMX2014]). Let $\phi : (X, \Delta) \rightarrow (X_g, \Delta_g)$ be a good minimal model (which exists by [Tan]). Let $f : X_g \rightarrow Z = \text{Proj } R(K_{X_g} + \Delta_g)$ the contraction (which exists by since $K_{X_g} + \Delta_g$ is nef, hence semiample by [Tan]). Then ϕ contracts the divisorial part of $\mathbb{B}(K_X + \Delta)$. Pick $t_0 > 0$ such that $(X_g, \Delta_g + t_0\phi_*A) = (X_g, \Delta_g + t_0A_g)$ is klt with A ample on X (note A_g is big, not in general nef). For H ample on X_g , let $\psi : X_g \rightarrow X'$ be the minimal model program with scaling, which terminates (again by [Tan]), and gives a minimal model of $(X_g, \Delta_g + t_0\Delta_g)$ over Z . For any curve contracted by f , $(K_{X_g} + \Delta_g) \cdot C = 0$, hence $K_{X'} + \psi_*\Delta_g = K_{X'} + \Delta' \equiv_Z 0$. Thus curves contracted by ψ have trivial intersection

with $K_{X_g} + \Delta_g$, and intersect negatively with A_g . Thus changing t does not affect which curves intersect negatively with $K_{X_g} + \Delta_g + tA_g$, and so X' is a minimal model of $(X_g, \Delta_g + tA_g)$ for all $t \in (0, t_0]$.

Now $\Delta' + t_0A'$ with $A' = \psi_*A_g$ (which is big) implies by the Theorem 8, that there exist only finitely many $K_{X'} + \Delta' + t_0A'$ negative extremal rays in $\overline{NE}(X')$. These are all necessarily just intersecting A' negatively, so decreasing t_0 , eventually we get to a point where a further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (Now we are on X' , not over Z). Again shrinking t_0 , suppose that $\psi \circ \phi$ is discrepancy negative w'r't $(X, \Delta + tA)$ for all t in the half open interval $(0, t_0]$. Note that $\mathbb{B}(K_X + \Delta + t_0A) \subset \mathbb{B}(K_X + \Delta)$, so ψ contracts only the curves not contracted by ϕ . Thus $\psi \circ \phi$ is discrepancy negative on the closed interval $[0, t_0]$, so X' is a minimal model of $(X, \Delta + tA)$ for all $t \in [0, t_0]$. Thus $\mathbb{B}(K_X + \Delta + tA)$ has the same divisorial components for all $t \in [0, t_0]$. Hence, any two minimal models for different $t \in [0, t_0]$ are birational and isomorphic in codimension one, and as the total dimension is two, they are thus isomorphic. \square

8. CONE, RATIONALITY, AND CONTRACTION / \mathbb{R}

Theorem 16. [CL, 3.5] *Let X be a normal scheme of relative dimension 2, proper over S , the spectrum of a DVR which has perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Let D_1, \dots, D_r be \mathbb{Q} -Cartier divisors on X . Assume that $R(X; D_1, \dots, D_r)$ is finitely generated, and let $D : \mathbb{R}^r \ni (\lambda_1, \dots, \lambda_r) \mapsto \sum \lambda_i D_i \in \text{Div}_{\mathbb{R}}(X)$ be the tautological map.*

(1) *The support of R is a rational polyhedral cone.*

(2) If $\text{Supp } R$ contains a big divisor and $D \in \sum \mathbb{R}_+ D_i$ is pseudoeffective, then $D \in \text{Supp } R$.

(3) There is a finite rational polyhedral subdivision $\text{Supp } R = \bigcup C_i$ such that σ_Γ is a linear function on C_i for every geometric valuation Γ of X . Furthermore, there is a coarsest subdivision with this property, in the sense that, if i and j are distinct, there is at least one geometric valuation Γ of X such that (the linear extensions of) $\sigma_\Gamma|_{C_i}$ and $\sigma_\Gamma|_{C_j}$ are different.

(4) There is a finite index subgroup $\mathbb{L} \subset \mathbb{Z}^r$ such that for all $n \in \mathbb{N}^r \cap \mathbb{L}$, if $D(n) \in \text{Supp } R$, then $\sigma_\Gamma(D(n)) = \text{mult}_\Gamma |D(n)|$.

Proof. (1, 2) follow easily from the cited theorem. For 3,4 note that [ELM, 4.7] is stated for an arbitrary Noetherian scheme, and this theorem is the only result used in the proof of this result which is given in [CL, 3.5]. \square

Theorem 17. ([CL, 3.6]) *Let (X, Δ) be a projective klt pair of dimension 2 (over a perfect field k) where Δ is big. If $K_X + \Delta$ is pseudoeffective, then it is \mathbb{Q} -effective.*

Proof. As in the source paper, but using Theorem 14 for the finite generation. \square

Theorem 18. *Let X be a normal projective variety of dimension 2 (over a perfect field k) and let D_1, \dots, D_r be \mathbb{Q} -Cartier \mathbb{Q} -divisors on X . Assume that the ring $R = R(X; D_1, \dots, D_r)$ is finitely generated, and let $\text{Supp } R = \bigcup C_i$ be a finite rational polyhedral subdivision such that for*

every geometric valuation Γ of X , σ_Γ is linear on C_i , as in Theorem 16. Denote $\varphi : \sum \mathbb{R}_+ D_i \rightarrow N^1(X)_\mathbb{R}$ the projection, and assume there exists k such that $C_k \cap \varphi^{-1}(\text{Amp } D) \neq \emptyset$. Then $C_k \subset \text{Supp } R \cap \varphi^{-1}(\text{Nef } X)$.

Proof. As in the source, but applying Theorem 16 and using abundance for this situation c.f. [Tan]. \square

Definition 19. [CL, def 4.1] Let W a finite dimensional real vector space, $C \subset W$ a closed convex cone spanning W , and $v \in W$. The visible boundary of C from v is

$$V = \{w \in \partial C \mid [v, w] \cap C = \{w\}\}.$$

Corollary 20. [CL, 4.2] (*Kawamata's Rationality, Cone and Contraction Theorem*) Let (X, Δ) be a klt pair of relative dimension 2, proper over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume either that $K_X + \Delta$ is \mathbb{Q} -Cartier and simple normal crossings over R , or is obtained after a finite number of steps of the minimal model program from such a \mathbb{Q} -divisor. Let V be the visible boundary of $\text{Nef}(X/R)$ from $v_0 \in N^1(X)_\mathbb{R}$ of the divisor $K_X + \Delta$. Then (1) every compact subset $F \subset \text{relint } V$ is contained in a union of finitely many supporting rational hyperplanes, and (2) every \mathbb{Q} -Cartier \mathbb{Q} -divisor on X with class in $\text{relint } V$ is semiample.

Proof. As in the cited theorem, noting it only requires the finite generation of Corollary 14, and Theorem 16. Part (2) is a consequence of Corollary 21. \square

9. BIG ABUNDANCE / \mathbb{R}

Let Γ a geometric valuation on X . Define

$$\sigma_\Gamma = \inf \{ \text{mult}_\Gamma D' \mid D \sim_{\mathbb{R}} D' \geq 0 \} \in \mathbb{R}.$$

As a result of Theorem 14, there is the following abundance theorem:

Corollary 21. [CL, 3.8] *Let (X, Δ) be a klt log smooth pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that Δ is big. If $K_X + \Delta$ is nef, then it is semi-ample.*

Proof. Let A an ample \mathbb{Q} -divisor on X such that $(X, \Delta + A)$ is klt. As Δ is big, it is possible to write $\Delta = \Delta' + A$ such that $(X, \Delta' + A)$ is klt. Thus the ring:

$$\mathcal{R} = R(K_X + \Delta, K_X + \Delta + A)$$

is finitely generated by Corollary 14, and $\text{Supp } \mathcal{R}$ contains a big divisor $(K_X + \Delta + A)$. Thus, any pseudo-effective

$$D \in \mathbb{R}_+(K_X + \Delta) + \mathbb{R}_+(K_X + \Delta + A)$$

is in $\text{Supp } \mathcal{R}$. By Theorem 16(2),

$$\text{Supp } \mathcal{R} = \mathbb{R}_+(K_X + \Delta, K_X + \Delta + A).$$

For any $\epsilon > 0$, and any geometric valuation Γ on X ,

$$\sigma_\Gamma(K_X + \Delta + \epsilon A) = 0$$

where, so that σ_Γ is zero on $\text{Supp } \mathcal{R}$, and thus the centre of Γ is not in $\mathbb{B}(K_X + \Delta)$ for any such Γ . \square

10. CONTRACTIONS GIVEN BY EXTREMAL RAYS / R

Lemma 22. [CL, 6.2] *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Suppose (X, Δ) is not nef, and A is a big \mathbb{Q} -divisor such that $(X, \Delta + A)$ is klt, and $K_X + \Delta + A$ is nef. Let λ be the nef threshold. Then $\lambda \in \mathbb{Q}^+$, and there is $R \subset \overline{NE}(X)$ with $(K_X + \Delta + \lambda A) \cdot R = 0$ and $(K_X + \Delta) \cdot R < 0$.*

Proof. As in the cited theorem, the only result being needed is the finite generation of Corollary 14. \square

11. BIG TERMINATION WITH SCALING

Theorem 23. *Let (X, Δ) be a projective \mathbb{Q} -factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Suppose Δ is big and simple normal crossings over R . Let A be a sufficiently general ample \mathbb{Q} -divisor on X such that $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is nef. Then the $K_X + \Delta$ -minimal model program with scaling of A terminates.*

Success is declared when there is a morphism $f : (X, \Delta) \rightarrow (X_n, \Delta_n)$ to a pair over a DVR such that on both geometric fibers $K_{X_n} + \Delta_n$ is nef. (Actually, applying [Liu, 5.3.24] and [Laz, 1.4.10], it suffices to show that the special fiber of the resulting model is nef).

Proof. This follows easily from [CL, 6.5], noting we have the required finite generation from Theorem 14. Let $\{\alpha_i\}$ be a sequence of numbers corresponding to a minimal model program defined on both fibers (α_i is the smallest positive real number such that $K_{X_i} + \Delta_i + \alpha_i A_i$ is nef on both fibers - although it suffices to check on the special fiber by Lemma 2). The claim is that the set of indices i is finite.

Choose r big divisors H_j arbitrarily close to $\Delta + \alpha$ for some norms on $N_1(X)$, and such that the *span* $\{K_X + H_j\}$ fills up all the dimensions of $N_1(X)$ and contains $K_X + \Delta + \alpha_1 A'$ on both fibers. Let C^1 be the \mathbb{R}_+ -cone spanned by the $K_{X_1} + H_i$. Let C^i denote the proper transform of C^1 , H_j^i the proper transforms of H_j , and let

$$\begin{aligned} R^i &= R(K_{X^i} + \Delta', K_{X^i} + H_1^i, \dots, K_{X^i} + H_r^i) \\ &\approx R^1 = R(K_X + \Delta, K_X + H_1, \dots, K_X + H_r) \end{aligned}$$

which are all finitely generated by Theorem 14 under the identification to R^1 .

By construction of the H^i , for each i , $G_i = K_{X_i} + \Delta'_i + \alpha_i A'_i \in \text{int } C^i$, and since G_i is nef, actually there are some ample divisors in $\text{int } C^i$. By Lemma 16, and since $\text{Supp } R^i = \bigcup C_\ell^i$, $\text{Supp } R^i = \bigcup C_\ell^i$ clearly contain big divisors, applying Theorem 17 (so that either G_i is \mathbb{Q} -Cartier so we can apply Theorem 16(3) or G_i is vacuously in $\text{Supp } R^i$) there are some ample divisors in C_ℓ^i for some ℓ . Thus using Theorem 18, the C_ℓ^i consist of the pullbacks of nef divisors from X_i . Since there are only finitely many cones in the subdivision $\text{Supp } R_1 = \bigcup C_\ell^1$, eventually there will exist an index i_0 , such that for $i > i_0$, the pullback of the nef cone

of X_i lands in a C_i^1 which is contained in the pullback of $Nef(X_{i'})$ for some $i' \leq i_0$. After this point, applying the negativity lemma (c.f. [Xu, 2.1] for positive characteristic and $\dim \leq 3$) as in [CL, 6.5] gives a morphism. \square

12. REDUCTIONS

Lemma 24. [HMX2014, 2.8.3] *Let (X, Δ) be a log smooth pair which is a scheme of dimension at most 3, with the coefficients of Δ belonging to $(0, 1]$, and with X projective. If (X, Δ) has a weak log canonical model, then there is a sequence $\pi : Y \rightarrow X$ of smooth blow ups of the strata of Δ such that if we write $K_Y + \Gamma = \pi^*(K_X + \Delta) + E$, where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$ and if we write*

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma(Y, K_Y + \Gamma),$$

then $\mathbb{B}_-(Y, K_Y + \Gamma')$ contains no strata of Γ' . If Δ is a \mathbb{Q} -divisor, then Γ' is a \mathbb{Q} -divisor.

Proof. Follows easily from the cited theorem. \square

Lemma 25. [GL, 2.3] *Let X/R a projective scheme of relative dimension 2 over a DVR R with algebraically closed residue field and perfect fraction field. Let (X, Δ) be a klt pair and $\varphi : W \rightarrow X$ a log resolution of (X, Δ) . Choose Δ_W so that $K_W + \Delta_W = \varphi^*(K_X + \Delta) + E$ with Δ_W and E effective \mathbb{Q} -Weil divisors with no common component. Let $F = \sum_{F_i \text{ } \varphi\text{-exceptional prime divisor}} F_i$ and $\Delta_W^\epsilon = \Delta_W + \epsilon F$, then (W, Δ_W^ϵ)*

is an ϵ -log smooth model of (X, Δ) . Then for $0 < \epsilon \ll 1$, a (good) minimal model for (W, Δ_W^ϵ) is also a (good) minimal model for (X, Δ) .

Proof. Follows easily from [BCHM, 3.6.10]. \square

Lemma 26. [HMX2014, 5.3] *Let X/R a projective scheme of relative dimension 2 over a DVR R , with algebraically closed residue field and perfect fraction field. Let (X, Δ) be a dlt pair with X \mathbb{Q} -factorial and projective and Δ a \mathbb{Q} -divisor. If Φ is a \mathbb{Q} -divisor such that*

$$0 \leq \Delta - \Phi \leq N_\sigma(X, K_X + \Delta),$$

then (X, Φ) has a (good) minimal model implies (X, Δ) has a (good) minimal model.

Proof. (Similar proof to the cited Theorem). Let $f : X \rightarrow Y$ the result of running a $(K_X + \Phi)$ minimal model program and let $Y \rightarrow Z$ be the ample model of $K_X + \Phi$. Let $\Delta_t = t\Delta + (1-t)\Phi$ so that if $0 < t \ll 0$, then f is also a $(K_X + \Delta_t)$ -MMP. If C generates a $K_X + \Delta_t$ -negative extremal ray, suppose that $(K_X + \Phi) \cdot C > 0$. Applying Theorem 8 on the special fiber,

$$-4 \leq (K_X + \Delta) \cdot C < 0.$$

Then, following the inequalities in [HMX2014, 5.1], gives a contradiction if $0 < t < \frac{1}{5}$. Thus every step of the $K_X + \Delta_t$ with scaling of an ample divisor is $(K_X + \Phi)$ -trivial, so after finitely many steps there is a model $g : X \rightarrow W$ contracting the components of $N_\sigma(X, K_X + \Delta_t)$ (this holds with no changes from [HMX2014, 2.7.1]).

Thus, as $\text{supp}(\Delta - \Phi) \subset \text{supp} N_\sigma(X, K_X + \Delta) = \text{supp} N_\sigma(X, K_X + \Delta_t)$, then $g_*(K_Y + \Phi)$ is semiample whenever $g_*(K_X + \Delta)$ is, and [HMX2014, 2.7.2] (which holds using only the resolution) implies that g is a minimal model of (X, Δ) , which is good when (X, Φ) has a good minimal model.

□

13. EXISTENCE OF MINIMAL MODELS SPECIAL CASE

Lemma 27. [HMX2010, 3.1] *Let (X, Δ) be a \mathbb{Q} -factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k) . Let $f : X \rightarrow Y$ be a step of the $(K_X + \Delta)$ -MMP. If f is birational and V is a non-canonical centre of (X, Δ) , then V is not contained in the indeterminacy locus of f , V_0 is not contained in the indeterminacy locus of f_0 , and the induced maps $\phi : V \rightarrow W$ and $\phi_k : V_k \rightarrow W_k$ are birational, where $W = f(V)$. Let $\Gamma = f_*\Delta$. Then $\mathbb{B}_-(Y_0, K_{Y_0} + \Gamma_0)$ contains no non-canonical centres of (Y_0, Γ_0) (so we may repeat the process), and if V is a non-klt centre or $V = X$, then $\phi : V \rightarrow W$ and $\phi_0 : V_0 \rightarrow W_0$ are birational contractions. If f is a Mori fibre space, then f_0 is not birational.*

Proof. (Follows easily from the cited Theorem). Suppose f is birational. Let V be a non-canonical centre of (X, Δ) . Let $g : X \rightarrow Z$ be the contraction of the extremal ray associated to f (so that $f = g$ unless f is a flip). Let $Q = g(V)$, and let $\psi : V \rightarrow Q$, be the induced morphism. As every component of V_k is a non-canonical centre

of (X_k, Δ_k) , then by hypothesis, components of V_k are not contained in $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$, thus ψ_k is defined at each such component, hence is birational. By upper-semicontinuity of the fibers of ψ , ψ is birational, and thus $\phi : V \rightarrow Q \rightarrow W$, and ϕ_k are both birational.

Now suppose V is a non-klt centre or $V = X$, the above holds in the first case, as non-klt centres are non-canonical. Comparing the discrepancies of the differentials of adjunction for Δ_k and Γ_k as in the cited proof shows that ϕ_k, f_k , and thus ϕ are birational contractions. On the other hand, if f is a Mori fibre space, then, as the dimension of the fibers of $f : X \rightarrow Y$ are upper-semicontinuous, f_k is not birational. \square

Theorem 28. [HMX2014, 3.2] *Let (X, Δ) be a \mathbb{Q} -factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k) and is log smooth. Let $f : X \rightarrow Y$ be a step of the $(K_X + \Delta)$ -MMP. Then the minimal model program with scaling terminates.*

Proof. (Similar proof to the cited theorem). Let $f : X \rightarrow Y$ the $(K_X + \Delta)$ -MMP with scaling of an ample divisor A . Let $\Gamma = f_*\Delta$ and $B = f_*A$. By construction, $K_Y + tB + \Gamma$ is nef for some $t > 0$. By Lemma 27, $f_k : X_k \rightarrow Y_k$ is a weak log-canonical model of $(X_k, tA_k + \Delta_k)$. If $K_X + \Delta$ is not pseudo-effective, then for $t > 0$, the result of f is a Mori fiber space, and by Lemma 27, Y_k is covered by $K_{Y_k} + tB_k + \Gamma_k$ -negative curves, contradicting bigness of $K_{X_k} + tA_k + \Delta_k$. Thus $K_X + \Delta$ is pseudo-effective, and given any $\epsilon > 0$, we may run the

MMP until $t < \epsilon$. Now we conclude by Theorem 15. Letting ϵ be the constant given in that theorem, the minimal models for $(X_k, tA_k + \Delta_k)$ are all isomorphic $t \in [0, \epsilon]$. Thus, once $t < \epsilon$, any more steps in the minimal model program with scaling must be an isomorphism on the special fiber, and thus an isomorphism. Thus there exists a minimal model (Y, Γ) for (X, Δ) . \square

14. EXISTENCE OF MINIMAL MODELS GENERAL CASE

Theorem 29. *Let (X, Δ) be a klt pair of relative dimension 2, proper over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume that $K_X + \Delta$ is \mathbb{Q} -Cartier, and log smooth over R . Then the minimal model of (X, Δ) exists.*

Proof. I start by repeating the reduction of [HMX2014, 6.1]. If necessary, extend R to be complete with algebraically closed residue field k , so the strata of Δ have irreducible fibers over R . Let $f_0 : Y_0 \rightarrow X_0$ be the birational morphism of Lemma 24. Under the log smooth hypothesis, and since the strata of Δ have irreducible fibers, and f_0 blows up strata of Δ_0 , extend f_0 to a birational morphism $f : Y \rightarrow X/R$ which is a composition of smooth blow ups of strata of Δ . Write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E$$

with $\Gamma \geq 0$ and $E \geq 0$, $f_*\Gamma = \Delta$, and $f_*E = 0$. Then (Y, Γ) is log smooth and the fibres of components of Γ are irreducible. By Lemma 25, (X, Δ) has a minimal model if (Y, Γ_ϵ) has a good minimal model,

where $(Y, \Gamma_\epsilon) = (Y, \Gamma + \epsilon F)$ is the ϵ -log smooth model of (X, Δ) with F the sum of f -exceptional divisors and $0 < \epsilon \ll 1$.

Replace (X, Δ) by (Y, Γ_ϵ) and set $\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma(X_k, K_{X_k} + \Delta_k)$ so that $\mathbb{B}_-(X_k, K_{X_k})$ contains no strata of Θ_k . Let $0 \leq \Theta \leq \Delta$ be the unique divisor such that $\Theta_k = \Theta|_{X_k}$. Let H relatively ample such that $(X, \Delta + H)$ is log smooth over R , and $kK_X + H$ is big. There is a commutative diagram:

$$\begin{array}{ccc} \pi_* \mathcal{O}_X(m(K_X + \Theta) + H) & \longrightarrow & \pi_* \mathcal{O}_X(m(K_X + \Delta) + H) \\ \downarrow & & \downarrow \\ H^0(X_k, \mathcal{O}_{X_k}(m(K_{X_k} + \Theta_k) + H_k)) & \longrightarrow & H^0(X_k, \mathcal{O}_{X_k}(m(K_{X_k} + \Delta_k) + H_k)) \end{array}$$

with surjective columns by Theorem 13, and with the bottom row an isomorphism. Applying Nakayama's lemma gives an isomorphism on the top row, so that $\Theta \geq \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)$. Again applying Theorem 13, gives $\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)$. Thus $\Delta - \Theta \leq N_\sigma(X, K_X + \Delta)$, so by Lemma 26, it suffices to find a minimal model for (X, Θ) . Replacing (X, Δ) by (X, Θ) , it suffices to assume that $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no strata of Δ_0 . Letting A be an ample divisor, we run the minimal model program with scaling of A . Now the assumptions of Theorem 28 apply, so we know that (X, Θ) has a minimal model.

□

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