MINIMAL MODELS FOR ARITHMETIC THREEFOLD GERMS

ANDREW EGBERT

ABSTRACT. In this note, I attempt to finish the log minimal model program proofs started in the SurfDIOP and Adjoint rings paper (see divisibility.wordpress.com). In SurfDIOP, finite generation of the canonical ring for klt, log smooth pairs on a scheme having two dimensional fibers over a DVR. In the syzygies and Adjoint rings note, the minimal model program for such (general type) pairs is studied (and the case that serre duality holds for X). In this, I relax the assumption on kodaira dimension. Most of this paper is just using results from characteristic 0, checking quickly if they hold over a mixed characteristic DVR, and then applying the finite generation in the big case (from [EGSurfDIOP]).

CONTENTS

1. Assumptions

In this paper, X will always be assumed to be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three.

2. Intersection Theory on X/R

Suppose X/R is a proper algebraic space of relative dimension $d=2$ with geometrically connected fibers where R is a discrete valuation ring with residue field k of characteristic $p>0$ and fraction field K of characteristic 0. (See for e.g. The compact Shimura Varieties mentioned from [Suh, 0.1].)

Lemma 1. [KU, 9.3] *Let $\varphi : X \to Spec(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $spec(R)$,* *where R is a DVR with algebraically closed residue field k, and field of fractions K. Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ. , if D, D' are divisors on X, then*

$$\begin{aligned} K^2_{X_k} &= K^2_{X_K} \\ (K_{X_k} \cdot D_k)_{X_k} &= (K_{X_K} \cdot D_K)_{X_K} \\ (D_k \cdot D'_k)_{X_k} &= (D_K \cdot D'_K)_{X_K} . \end{aligned}$$

If X/R is smooth, then lemma 1 applies to show that for any divisors C, D extending to both fibers, we can just define $C.D = (C|_{X_k}.D|_{X_k})_{X_k}$. On the other hand, if X/R is merely normal and proper, but is actually a scheme, the resolution of singularities, theorem 3, holds. Thus the intersection theory can be defined as in [Tan, Def 3.1]: $f : X' \to X$ is a resolution, and $C.D = f^*C.f^*D$ for two divisors C, D on Y/R. By properness, this intersection extends by linearity to Weil divisors with $\mathbb{Q}$ or $\mathbb{R}$ coefficients. Numerical equivalence and $N^1(X)_{\mathbb{Q},\mathbb{R}}$ are then defined as usual.

Note the two different hypothesis here: The invariance of plurigenera theorem requires a proper, smooth algebraic space of relative dimension 2, while the existence of minimal models requires the intermediate steps of the minimal model program to have some intersection theory, and thus be schemes (but not necessarily smooth). Recall that a Cartier divisor C on a variety is usually called nef if $C.D \geq 0$ for all curves C.

Lemma 2. *Let S be an affine Dedekind scheme and $f : X \to S$ a projective morphism. Let $\mathcal{L}$ be an invertible sheaf on X such that $\mathcal{L}_s$ is nef for every closed point $s \in S$. Then $\mathcal{L}$ is nef.*

Proof. By [Liu, 5.3.24], the statement holds when nef is replaced by ample. Now, c.f. [Laz, 1.4.10], it suffices to choose an ample divisor H which restricts to X_s, so that $D_s + \epsilon H_s$ is ample for all sufficiently small ϵ. Then $D + \epsilon H$ is ample for all sufficiently small ϵ, and so D is nef. □

3. Resolution

Theorem 3. [CP2014, 1.1] *Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*

(i) X' is everywhere regular

(ii) π induces an isomorphism $\pi^{-1}(Reg\ X) \approx Reg\ X$

(iii) $\pi^{-1}(Sing\ X)$ is a strict normal crossings divisor on X'.

4. Effective KLT Kawamata Viehweg Vanishing

In this section, I note that a simple modification of Tanaka's vanishing theorem [Tan2] gives a version of the (log) Kawamata Viehweg Vanishing in which the multiple of the nef divisor is at least computable. Note that there is a similar recent result of [DF], which is more restrictive in that it requires a smooth variety rather than a normal projective klt pair.

Theorem 4. [Ter] *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X. Assume that either*

(1) $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or

(2) X is of general type with

$p \geq 3$ *and* $(D^2) > vol\,(X)$ *or*

$p = 2$ *and* $(D^2) > max\,\{vol\,(X)\,, vol\,(X) - 3\chi\,(\mathcal{O}_X) + 2\}\,.$

Then

$$H^i\,(X, \mathcal{O}_X\,(K_X + D)) = 0$$

for all $i > 0$.

Theorem 5. *(Effective KLT Kawamata-Viehweg Vanishing for normal surfaces) Let* (X, Δ) *be a normal projective klt surface (over an algebraically closed field of characteristic* $p > 2$*) where* Δ *is an effective* $\mathbb{R}$*-divisor. Let* N *be a nef and big* $\mathbb{R}$*-cartier* $\mathbb{R}$*-divisor with* $N^2 > vol\,(X)$*. Let* D *be a* $\mathbb{Q}$*-Cartier* $\mathbb{Z}$*-divisor such that* $D - (K_X - \Delta)$ *is nef and big. Then* $H^i\,(X, \mathcal{O}_X\,(D + rN + N')) = 0$ *for every* $i > 0$*, every positive real number* $r \geq 1$*, and every nef* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* N' *such that* $rN + N'$ *is a Cartier divisor.*

Proof. Verbatim from [Tan2, 2.11] except that Weak Kodaira Vanishing [Tan2, 2.4] in the proof of [Tan2, 2.6] is replaced by Terakawa's vanishing theorem 4.

□

5. Invariance of Plurigenera Special Case

Lemma 6. [KU, 9.4] *Let* $\varphi : X \to Spec\,(R)$ *be an algebraic two-dimensional space, proper, separated, and of finite type over* $spec\,(R)$*, where* R *is a DVR with algebraically closed residue field* k*, and field of fractions* K*. Let* X_K *(resp.* X_k*) denote the generic geometric (resp.*

closed) fibre of φ. If X_k contains an exceptional curve of the first kind e, there exists a DVR $\tilde{R} \supset R$, with residue field isomorphic to k, and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \to \tilde{Y}$ over $Spec\left(\tilde{R}\right)$ such that on the closed fibre, π induces the contraction of the exceptional curve e. Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.

Proof. By [A1, Cor 6.2] $Hilb_{X/Spec(R)}$ is represented by an algebraic space $\mathscr{H}$ which is locally of finite type over $Spec\,(R)$. Let Y be the irreducible component containing the point $\{e\}$ corresponding to the exceptional curve e on the special fiber. Then $e \approx \mathbb{P}^1_k$ and $N_{e/X} \approx \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, so Y is regular at $\{e\}$ and of dimension 1. Since e is fixed in the special fiber, the structure morphism $p : Y \to Spec\,(R)$ is surjective. By [Bo, 2.2.14][1], we can find an étale cover $\tilde{R} \supset R$ and a morphism $j : Spec\left(\tilde{R}\right) \to Y$ over $Spec\,(R)$ with $j\,(\tilde{o}) = \{e\}$ (if R is not already complete, then first extend R to a complete DVR using [Gr, 0.6.8.2,3] so that $\tilde{R}$ is again a DVR). As k is assumed algebraically closed, and $\tilde{R} \to R$ is unramified, then the extension of residue fields is finite and separable at the closed point of R, and hence an isomorphism of residue fields. Let $\hat{p} : \mathscr{E} \to Spec\left(\tilde{R}\right)$ be the pull-back of the universal family over Y. As the closed fibre $\mathscr{E}_0$ is a projective line,

---

1

Remark 7. if k is perfect, the proof of the cited lemma goes seems to go through when $f : X \to S$ is merely smooth at $x \in X$ and $f : X \to S$ is locally of finite type.

we may choose the morphism j in such a way that the generic fibre $\mathscr{E}_1$ of $\hat{p}$ is also a projective line. Moreover, $\mathscr{E}$ can be considered as a smooth closed algebraic subspace of codimension 1 in $\hat{X} = X \otimes \tilde{R}$. By lemma 1, $-1 = \mathscr{E}_0^2 = \mathscr{E}_1^2$. Hence $\mathscr{E}_1$ is also an exceptional curve of the first kind. Hence by [A2, Cor 6.11] there exists a contraction morphism $\pi : \hat{X} \to \hat{X}$ over $Spec\left(\tilde{R}\right)$ which contracts $\mathscr{E}$ to a section of $\tilde{\varphi} : \tilde{X} \to Spec\,(R)$ where $\tilde{\varphi}$ is proper, smooth, separated and of finite type over $Spec\left(\tilde{R}\right)$.

□

Lemma 8. [KK94, 2.3.5] *Let (S, B) be a log-canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C.\,(K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

Theorem 9. [Tan, 4.4] *Let X be a projective normal surface and let Δ be an effective $\mathbb{R}$-divisor such that $K_X+\Delta$ is $\mathbb{R}$-cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let H be an $\mathbb{R}$-cartier ample $\mathbb{R}$-divisor. Then the following assertions hold:*

(1) $\overline{NE}\,(X) = \overline{NE}\,(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}\,[C_i]$

(2) $\overline{NE}\,(X) = \overline{NE}\,(X)_{K_X+\Delta+H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}\,[C_i]$

(3) Each C_i in (1) and (2) is rational or $C_i = B_j$ for some B_j with $B_j^2 < 0$.

Lemma 10. [Tan, 5.3] *Let X be a projective normal surface and let C be a curve in X such that $r\,(K_X + C)$ is Cartier for some positive integer r.*

Theorem 11. *(1) If $C\cdot(K_X+C)<0$, then $C\approx\mathbb{P}^1$.*

(2) If $C\cdot(K_X+C)=0$, then $C\approx\mathbb{P}^1$ or $\mathcal{O}_C\left((K_X+C)^{[r]}\right)\approx\mathcal{O}_C$.

Proposition 12. *Let (X,Δ) be a terminal log smooth pair of relative dimension 2 over a DVR R with algebraically closed residue field k of characteristic $p>2$ and perfect fraction field K. Assume that $K_X+\Delta$ is $\mathbb{Q}$-Cartier and $\nu(K_X+\Delta)\neq 1$. Assume that $\mathbb{B}_-(K_{X_k}+\Delta_k)\wedge\Delta_k=\emptyset$. Then there exists an m_0 such that for $m\in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Delta_K\right)\right).$$

If $\nu(K_X+\Delta)=0$ or 2, then m_0 is computable. If serre duality holds on X, and X/R is unramified, then the same holds for $\nu(K_X+\Delta)=1$.

The proof of proposition 12, is given in the following claims.

Claim. Assumptions as above, after passing to an extension R' of R, there is a proper, smooth algebraic space X^{min}/R' and an R' morphism $X\otimes R'\to X^{min}$ such that both $X_k\to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}}+\Delta_{X_k^{min}}$ is nef.

Proof. As k is algebraically closed of characteristic $p>0$, then by the Cone Theorem 9,

$$\overline{NE}(X_k)=\overline{NE}(X_k)_{K_{X_k}+\Delta_k\geq 0}+\sum\mathbb{R}_{\geq 0}[C_i]$$

with each C_i is rational or $C_i=B_j$ for some B_j a component of Δ with $B_j^2<0$. Under the assumption that $\mathbb{B}_-(K_{X_k}+\Delta_k)\wedge\Delta_k=\emptyset$, then actually each C_i is rational and is not a component of Δ_k. Thus

$C_i.\Delta_k \geq 0$ so $C_i.K_{X_k} < 0$, and since $N_\sigma\left(K_{X_k} + \Delta_k\right)$ is a $\mathbb{Q}$-divisor by [Tan, 7.1], then Theorem 10 implies $C_i \approx \mathbb{P}^1$. By Lemma 8, C_i can be contracted to a log-canonical point, which is actually a smooth point under the terminal assumption. Thus C_i is an exceptional curve of the first kind, so it is possible to apply Lemma 6.

By Lemma 6, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \to \tilde{X}$ induces the contraction of C_i on X_k and X_K, and further $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since k is algebraically closed, it induces identity on residue fields. Now I need to work with $\tilde{X}$. In order to apply the Cone Theorem 9 again, I need $\tilde{X}_k$ to be projective, but a smooth algebraic space of dimension 2, proper, separated, and of finite type over an algebraically closed field is projective, so $\tilde{X}_k$ is projective. Thus the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no problem in extending R. As the Picard number drops each step, there are only finitely many steps. □

Claim 13. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef. (In the general type case, there is another argument, c.f. two claims down.)

Proof. It suffices to apply 2 to X_k^{min}, and then restrict to X_K. □

Now we proceed by cases depending on the Kodaira dimension.

Claim 14. In the case that (X_k, Δ_k) is big or has $\nu = 0$, there is a computable m_0 such that for $m_0|m$, we have

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Proof. Suppose first that (X_k, Δ_k) is big. By the above, we achieve X^{min} such that $K_X + \Delta$ is nef on both X_k^{min} and X_K^{min}. By the Kawamata Viehweg Vanishing Theorem 5 applied to the special fiber (and the semicontinuity theorem) there exists a computable $m_0 \gg 0$ such that for $m_0 < m$ and $i > 0$, we have

$$h^i\left(m\left(K_{X_K} + \Delta_K\right)\right) = h^i\left(m\left(K_{X_k} + \Delta_k\right)\right) = 0.$$

In fact, if $\Delta = 0$, then Ekedahl's vanishing Theorem [Ek] can be applied. Thus, applying the invariance of Euler characteristic [Liu, 5.3.22], and birational invariance of the plurigenera, it follows that for $m > m_0$,

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Now suppose that $\nu\left(K_{X_k} + \Delta_k\right) = 0$. In the case that the special fiber has $\nu\left(K_{X_k} + \Delta_k\right) = 0$, there is a computable m_0 such that for $m_0|m$, we have

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

As before, we achieve an $\pi' : X^{min} \to R'$ and an R' morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and both $K_{X_k^{min}} + \Delta_{X_k^{min}}$ and $K_{X_K^{min}} + \Delta_{X_K^{min}}$ are nef.

Applying [Tan], we find that $K_{X_k^{min}}+\Delta_{X_k^{min}}$ is semi-ample. Under the pseudo-effective assumption, and since $\nu\left(K_{X_k^{min}}+\Delta_{X_k^{min}}\right)=0$, but $K_{X_k^{min}}+\Delta_{X_k^{min}}$ is nef, then actually $K_{X_k^{min}}+\Delta_{X_k^{min}}\equiv 0$. A high multiple of a semi-ample, and numerically trivial bundle is globally generated and numerically trivial, hence $\mathcal{O}_{X_k^{min}}\left(m\left(K_{X_k^{min}}+\Delta_{X_k^{min}}\right)\right)=\mathcal{O}_{X_k^{min}}$, $m_k\gg 0$. Actually the same holds on both fibers, since the abundance has been proven over a field k. Thus, perhaps taking $m'=m_k\cdot m_K$, then the conclusion follows cohomological flatness of the smooth map $X^{min}\to R'$, c.f. [Liu, Exc. 5.3.14]. □

Finally, suppose that $\nu\left(K_X+\Delta\right)=1$ (this part of the article I am not sure of, but it is not necessary for the rest of this article), Serre duality holds on X, and X/R is unramified.

By the previous claim, we achieve an $\pi':X^{min}\to R'$ and an R' morphism $X\otimes R'\to X^{min}$ such that both $X_k\to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and both $K_{X_k^{min}}+\Delta_{X_k^{min}}$ and $K_{X_K^{min}}+\Delta_{X_K^{min}}$ are nef. Now, by [Liu, 5.3.20] it suffices to show that $H^1\left(X,\mathcal{O}_X\left(m\left(K_X+\Delta\right)\right)\right)$ is torsion free or that $R^1\pi'_*\mathcal{O}_X\left(m\left(K_X+\Delta\right)\right)$ is torsion free.

C.f. [Suh, 1.2.1] as R' is unramified, X_k^{min} lifts to W_2. C.f. [Tan], $K_X+\Delta$ is semiample, so $\left|m\left(K_{X_k}+\Delta_k\right)\right|$ contains a smooth curve for $m\gg 0$ which therefore lifts to $W_2(k)$. Let $L:=m\left(K_X+\Delta\right)$. As we are in the terminal category, and L has no stable base locus, then point blowups preserve our $W_2(k)$ lifting of X_k, and also global generation is preserved, c.f. [Suh, Lie]. Thus the reductions of [EV, 5.12.a] hold in this setting. As we are assuming Serre Duality holds, the proof of torsion freeness from [Ko2, 10.15] holds, and thus, as L is semi-ample, $R^1\pi'_*\mathcal{O}_X(m(K_X+\Delta))$ is torsion-free.

6. Invariance of Plurigenera and Finite Generation

Theorem 15. *Let (X,Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p>0$ and perfect fraction field K. Assume that $K_X+\Delta$ is, big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists a (computable) m_0 depending on the intersection numbers, such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0(m(K_{X_k}+\Delta_k)) = h^0(m(K_{X_K}+\Delta_K))$$

thus, after possibly extending R, the canonical ring $R(K_X+\Delta)$ is finitely generated over R. If $K_X+\Delta$ is pseudo-effective(not necessarily big), a contraction of a $K_{X_k}+\Delta_k$-negative extremal ray cut out by an ample H on R induces a contraction of a $K_X+\Delta$ negative extremal ray. If the contraction is a flipping contraction, then the flip exists.

Proof. Since the hypothesis and conclusion are preserved by base change, then extending R, we may assume that k is algebraically closed and that R is complete. The proof follows similarly to [HMX2010, 1.6] except that Proposition 12 is used in place of Kawamata Viehweg Vanishing. Recall that the log-canonical ring of $K_{X_k}+\Delta_k$ is finitely generated (c.f. [Tan, 7.1]) so $N_\sigma(K_{X_k}+\Delta_k)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma(K_{X_k}+\Delta_k)$$

and let $0 \leq \Theta \leq \Delta$ be a $\mathbb{Q}$-divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. Replacing (X, Δ) by a blow-up, assume (X, Δ) is terminal. By definition of N_σ, letting $m \gg 0$ to fit the hypothesis of Proposition 12, and sufficiently divisible such that $m\left(K_{X_k} + \Theta_k\right)$ is integral, then

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_k} + \Theta_k\right)\right)$$

and furthermore, by Proposition 12,

$$h^0\left(m\left(K_{X_k} + \Theta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Theta_K\right)\right).$$

As $\Theta \leq \Delta$, the first part of the theorem follows by semicontinuity.

Now by [Tan], there is $m \gg 0$, such that

$$\bigoplus_{m \geq 0} H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right)$$

is generated in degree 1, so that there is a surjection

$$S^l H^0\left(m\left(K_{X_k} + \Delta_k\right)\right) \twoheadrightarrow H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right).$$

Consider the following diagram:

$$\begin{array}{ccc} S^l H^0\left(m\left(K_X + \Delta\right)/R\right) & \xrightarrow{\alpha} & H^0\left(ml\left(K_X + \Delta\right)/R\right) \\ \downarrow & & \downarrow \\ S^l H^0\left(m\left(K_{X_k} + \Delta_k\right)\right) & \xrightarrow{\twoheadrightarrow} & H^0\left(ml\left(K_{X_k} + \Delta_k\right)\right) \end{array}$$

The vertical maps are surjective by the first part of the theorem (and applying Nakayama's lemma to pull back the generators of the k-modules). Now again using, Nakayama's Lemma [GH, Chapter 5.3], surjectivity of the bottom map implies surjectivity of α, and thus $R(K_X+\Delta)$ is finitely generated over R.

Finally suppose that there is a contraction of a curve on the closed fiber (X_k,Δ_k) corresponding to a $K_{X_k}+\Delta_k$ -negative extremal ray Σ_k. If necessary, extend R to be complete and k algebraically closed. Let $H_k = H|_{X_k}$ be such that $(K_{X_k}+\Delta_k+H_k).\Sigma_k = 0$. The goal is to show that

$$R(K_X+\Delta+H/R)$$

is finitely generated so that there exists a contraction defined by $X_1 = Proj\ R(K_X+\Delta+H/R)$. However, as H is ample, then under the assumption that $K_{X_k}+\Delta_k$ is pseudo-effective, $K_{X_k}+\Delta_k+H_k$ is actually big, and so this follows. □

7. Finite Generation (Big Case)

Corollary 16. *Let (X,Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p>0$ and perfect fraction field K . Assume that $K_X+\Delta$ is $\mathbb{Q}$-Cartier, and simple normal crossings over R. If $A_1,...,A_m$ are any ample divisors, then (1) the ring $R(X,K_X+\Delta+A_1,...,K_X+\Delta+A_n)$ is finitely generated over R.*

Proof. C.f. the proof of [CZ, 2.11], it suffices to prove that if $G=\sum_{i=1}^k m_i(a(K_X+\Delta_i+A_i))$, fixed $a\in\mathbb{N}$, and $m_1,...,m_k\geq 0$ and $l\in\{1,...,l\}$ then there exists m_l such that the following maps are surjective

$$H^0(X,\mathcal{O}_X(G-a(K_X+\Delta-A_l)))\otimes H^0(X,\mathcal{O}_X(a(K_X+\Delta+A_l)))$$

$$\to H^0(X,\mathcal{O}_X(G))$$

is surjective. By Nakayama's lemma applied as above, it suffices to show the surjectivity on the central fiber.

Applying [Tan], there exists a minimal model (X'_k, Δ'_k) for (X_k, Δ_k), such that $K_{X_k} + \Delta_k$ is nef and semi-ample. Thus there is $m \gg 0$ such that $m\,(K_{X_k} + \Delta_k)$ is both bpf and $m-2$ is large enough so the effective Kawamata Viehweg vanishing Theorem 5applies. Now the proof of [CZ, 2.7] applies. □

8. Finiteness of Models For Surfaces

Theorem 17. *Let X be a two dimensional normal variety over a perfect field of characteristic $p > 0$. Let (X, Δ) be a log smooth KLT pair and A an ample $\mathbb{Q}$-divisor on X. Then there exists an $\epsilon > 0$, such that the minimal models (and the output of the minimal model program with scaling) for $(X, \Delta + tA)$ are all isomorphic for $t \in [0, \epsilon]$.*

Proof. (Essentially same as [Lai, 26,27] - I'm not sure if this is the correct cite for Lai's thesis which is what I am intending to cite, as it appears to be a springerlink thingy - in any case the proof is essentially the same or simplified, but restated for the slightly different setting, and simplifies due to the fact that birational isomorphism in codimension 1 on a surface gives an isomorphism There is also a proof of essentially the same result in [HMX2014]). Let $\phi : (X, \Delta) \to (X_g, \Delta_g)$ a good minimal model (exists by [Tan]). Let $f : X_g \to Z = Proj\ R\left(K_{X_g} + \Delta_g\right)$ the contraction (exists by since $K_{X_g} + \Delta_g$ is neff, hence semiample by [Tan]). Then ϕ contracts the divisorial part of $\mathbb{B}\,(K_X + \Delta)$. Pick $t_0 > 0$ such that $(X_g, \Delta_g + t_0\phi_* A) = (X_g, \Delta_g + t_0 A_g)$ is klt with A ample on X

(note A_g is big, not in general nef). For H ample on X_g, let $\phi : X_g \to X'$ be the minimal model program with scaling, which terminates again by [Tan], and gives a minimal model of $(X_g, \Delta_g + t_0\Delta_g)$ over Z. For any curve contracted by f, $\left(K_{X_g} + \Delta_g\right).C = 0$, hence $K_{X'} + \psi_*\Delta_g = K_{X'} + \Delta' \equiv_Z 0$. Thus curves contracted by ψ have trivial intersection with $K_{X_g} + \Delta_g$, and intersect negatively with A_g. Thus changing t doesn't affect which curves intersect negatively with $K_{X_g} + \Delta_g + tA_g$, and so X' is a minimal model of $(X_g, \Delta_g + tA_g)$ for all $t \in (0, t_0]$.

Now $\Delta' + t_0A'$, $A' = \psi_*A_g$ is big implies by the cone theorem (again found in [Tan]), there exist only finitely many $K_{X'} + \Delta' + t_0A'$ negative extremal rays in $\overline{NE}\left(X'\right)$. These are all necessarily just intersecting A' negatively, so decreasing t_0, eventually we get to a point where further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (Now we are on X', not over Z). Again shrinking t_0, suppose that $\psi \circ \phi$ is discrepancy negative w'r't $(X, \Delta + tA)$ for all t in the half open interval $(0, t_0]$. Note that $\mathbb{B}\left(K_X + \Delta + t_0A\right) \subset \mathbb{B}\left(K_X + \Delta\right)$, so ψ contracts only the things not contracted by ϕ (which is no divisors). Thus $\psi \circ \phi$ is discrepancy negative on the closed interval $[0, t_0]$, so X' is a minimal model of $(X, \Delta + tA)$ for all $t \in [0, t_0]$. Thus $\mathbb{B}\left(K_X + \Delta + tA\right)$ has hte same divisorial components for all $t \in [0, t_0]$. So any two minimal models for different $t \in [0, t_0]$ are isomorphic in codimension one, and as the total dimension is two, they are thus isomorphic. □

9. Cone, Rationality, and contraction / R

Theorem 18. *([CL, 3.5] Let X be a normal scheme of relative dimension 2, proper over S, the spectrum of a DVR which has perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier divisors on X. Assume that $R(X; D_1, ..., D_r)$ is finitely generated, and let $D : \mathbb{R}^r \ni (\lambda_1, ..., \lambda_r) \mapsto \sum \lambda_i D_i \in Div_{\mathbb{R}}(X)$ be the tautological map.*

(1) The support of R is a rational polyhedral cone.

(2) If Supp R contains a big divisor and $D \in \sum \mathbb{R}_+ D_i$ is pseudoeffective, then $D \in$ Supp R.

(3) There is a finite rational polyhedral subdivision Supp $R = \bigcup C_i$ such that σ_Γ is a linear function on C_i for every geometric valuation Γ of X. Furthermore, there is a coarsest subdivision with this property, in the sense that, if i and j are distinct, there is at least one geometric valuation Γ of X such that (the linear extensions of) $\sigma_\Gamma|_{C_i}$ and $\sigma_\Gamma|_{C_j}$ are different.

(4) There is a finite index subgroup $\mathbb{L} \subset \mathbb{Z}^r$ such that for all $n \in \mathbb{N}^r \cap \mathbb{L}$, if $D(n) \in$ Supp R, then $\sigma_\Gamma(D(n)) = mult_\Gamma |D(n)|$.

Proof. 3.5(1, 2) hold the same as in the source paper. For $3, 4$ note that [ELM, 4.7] is stated for an arbitrary Noetherian scheme, and this theorem is the only result used in the proof of this result which is given in [CL, 3.5].

□

Theorem 19. *([CL, 3.6]) Let (X, Δ) be a projective klt pair of dimension 2 (over a perfect field k) where Δ is big. If $K_X + \Delta$ is pseudoeffective, then it is $\mathbb{Q}$-effective.*

Proof. As in the source paper, but using Theorem 16 for the finite generation.

∎

Definition 20. [CL, def 4.1] Let W a finite dimensional real vector space, $C \subset W$ a closed convex cone spanning W, and $v \in W$. The visible boundary of C from v is

$$V = \{w \in \partial C \mid [v, w] \cap C = \{w\}\}.$$

Corollary 21. [CL, 4.2] *(Kawamata's Rationality, Cone and Contraction Theorem) Let (X, Δ) be a klt pair of relative dimension 2, proper over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K . Assume either that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and simple normal crossings over R, or is obtained after a finite number of steps of the minimal model program from such a $\mathbb{Q}$-divisor. Let V be the visible boundary of $Nef\,(X/R)$ from $v_0 \in N^1(X)_{\mathbb{R}}$ of the divisor $K_X + \Delta$. Then (1) every compact subset $F \subset relint\ V$ is contained in a union of finitely many supporting rational hyperplanes, and (2) every $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X with class in relint V is semiample.*

Proof. As in the cited theorem, noting it only requires the finite generation of Corollary 16, and Theorem 18. Part (2) is a consequence of Corollary 22.

□

10. Big Abundance / R

Let Γ a geometric valuation on X. Define

$$\sigma_\Gamma = inf\left\{mult_\Gamma D' \mid D \sim_{\mathbb{R}} D' \geq 0\right\} \in \mathbb{R}$$

. As a result of the generalized finite generation:

Corollary 22. [CL, 3.8]*Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that Δ is big. If $K_X + \Delta$ is nef, then it is semi-ample.*

Proof. Let A an ample $\mathbb{Q}$-divisor on X such that $(X, \Delta + A)$ is klt. As Δ is big, it is possible to write $\Delta = \Delta' + A$ such that $(X, \Delta' + A)$ is klt. Thus the ring:

$$\mathcal{R} = R\left(K_X + \Delta, K_X + \Delta + A\right)$$

is finitely generated by Corollary 16, and *Supp* $\mathcal{R}$ contains a big divisor $(K_X + \Delta + A)$, so any pseudo-effective

$$D \in \mathbb{R}_+\left(K_X + \Delta\right) + \mathbb{R}_+\left(K_X + \Delta + A\right)$$

is in $Supp\ \mathcal{R}$. By Theorem 18(2),

$$Supp\ \mathcal{R} = \mathbb{R}_+ (K_X + \Delta, K_X + \Delta + A).$$

For any $\epsilon > 0$, and any geometric valuation Γ on X, $\sigma_\Gamma (K_X + \Delta + \epsilon A) = 0$ where, so that σ_Γ is zero on $Supp\ \mathcal{R}$, and thus the centre of Γ is not in $\mathbb{B}(K_X + \Delta)$ for any such Γ. □

11. Contractions given by Extremal Rays / R

Lemma 23. [CL, 6.2]*Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Suppose (X, Δ) is not nef, and A is a big $\mathbb{Q}$-divisor such that $(X, \Delta + A)$ is klt, and $K_X + \Delta + A$ is nef. Let λ be the nef threshold. Then $\lambda \in \mathbb{Q}^+$, and there is $R \subset \overline{NE}(X)$ with $(K_X + \Delta + \lambda A).R = 0$ and $(K_X + \Delta).R < 0$.*

Proof. As in the cited theorem, the only result being needed is the finite generation Corollary 16. □

12. Big Termination With Scaling

Theorem 24. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Suppose Δ is big and simple normal crossings over R. Let A be sufficiently general ample $\mathbb{Q}$-divisor on X such that $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is nef. The minimal model program with scaling of A terminates.*

Success is declared when there is a morphism $f:(X,\Delta)\to(X_n,\Delta_n)$ to a pair over a DVR such that on both geometric fibers $K_{X_n}+\Delta_n$ is nef. (Actually, applying [Liu, 5.3.24] and [Laz, 1.4.10], it suffices to show that the special fiber of the resulting model is nef).

Proof. The argument follows [CL, 6.5]. Let $\{\alpha_i\}$ be a sequence of numbers corresponding to a minimal model program defined on both fibers (α_i is the smallest positive real number such that $K_{X_i}+\Delta_i+\alpha_i A_i$ is nef on both fibers - although it suffices to check on the special fiber by Lemma 2). The claim is that the set of indices i is finite.

Choose r big divisors H_j arbitrarily close to $\Delta+\alpha$ for some norms on $N_1(X_k), N_1(X_K)$, and such that the *span* $\{K_{X_1}+H_j\}$ fills up all the dimensions of $N_1(X_k)$ and $N_1(X_K)$ simultaneously and contains $K_X+\Delta+\alpha_1 A'$ on both fibers. Let C^1_k, C^1_K be the $\mathbb{R}_+$-cone spanned by the $K_{X_1}+H_i$ on X_k, X_K respectively. Let C^i_k, C^i_K denote the proper transforms, $H_j^{i,k}$ or $H_j^{i,K}$ the proper transforms of $H_j|_{X_k}$, $H_j|_{X_K}$ and let

$$R^1_k \approx R^i_k = R\left(K_{X^i_k}+\Delta'_k, K_{X^i_k}+H_1^{i,k},...,K_{X^i_k}+H_r^{i,k}\right),$$
$$R^1_K \approx R^i_K = R\left(K_{X^i_K}+\Delta'_K, K_{X^i_K}+H_1^{i,K},...,K_{X^i_K}+H_r^{i,K}\right).$$

which are all finitely generated under the identification to R^1_k, R^1_K as in Theorem 16.

By construction of the H^i, for each i, $G_i=K_{X_i}+\Delta'_i+\alpha_i A'_i\in$ *int* C^i on both fibers, and since G_i is nef on both fibers, actually there are some ample divisors in *int* C^i. By Lemma 18, and since *Supp* $R^i_k=\bigcup C^i_{\ell_k}$, *Supp* $R^i_K=\bigcup C^i_{\ell_K}$ clearly contain big divisors, applying Theorem 19 (so that either G_i is $\mathbb{Q}$-Cartier so we can apply 3.5(3) or G_i is vacuously in $Supp\ R^i_{K,k}$) there are some ample divisors in $C^i_{\ell_k}$, $C^i_{\ell_K}$ for some ℓ_k,ℓ_K. Thus using Theorem 18, $C^i_{\ell_k}$, $C^i_{\ell_K}$ consist of the pullbacks of nef divisors from X_i. Focusing on X_k, since there are only finitely many cones in the subdivision $Supp\ R_1 = \bigcup C^1_{\ell_k}$, eventually there will exist an index i_0, such that for $i > i_0$, the pullback of the nef cone of X_i lands in a $C^1_{\ell_k}$ which is contained in the pullback of $Nef\,(X_{i'})$ for some $i' \leq i_0$. After this point, applying the negativity lemma (c.f. [Xu, 2.1] for positive characteristic and $dim \leq 3$) as in [CL, 6.5] gives a morphism on the special fiber. By construction of the above minimal model program with scaling, then for $i > i_0$, and applying the same logic to X^i_K, gives that eventually these are all also morphisms. □

13. Reductions

Lemma 25. [HMX2014, 2.8.3] *Let (X, Δ) be a log smooth pair which is a scheme of dimension at most 3 (or wherever resolution holds) where the coefficients of Δ belong to $(0, 1]$ and X is projective. If (X, Δ) has a weak log canonical model, then there is a sequence $\pi : Y \to X$ of smooth blow ups of the strata of Δ such that if we write $K_Y + \Gamma = \pi^*(K_X + \Delta) + E$, where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$ and if we write*

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma\,(Y, K_Y + \Gamma)\,,$$

then $\mathbb{B}_-\,(Y, K_Y + \Gamma')$ contains no strata of Γ'. If Δ is a $\mathbb{Q}$-divisor, then Γ' is a $\mathbb{Q}$-divisor.

Proof. (same as source which is stated over $\mathbb{C}$, since just using resolution). □

Lemma 26. [HX, 2.10] *Let X/R a projective scheme of relative dimension 2 over a dvr R, with algebraically closed residue field and perfect fraction field. Let(X,Δ) a dlt pair and $\mu : X' \to X$ a proper birational morphism. Write $K_{X'}+\Delta' = \mu^*(K_X+\Delta)+F$ where Δ' and F are effective with no common components. Then (X,Δ) has a good minimal model over U if and only if (X',Δ') has a good minimal model over U.*

Proof. Same as in the source, noting that resolution holds in this dimension, and the termination for the big case holds by Theorem 24, and termination in the abundant case holds by Theorem 32. □

Lemma 27. [HMX2014, 5.3]*Let X/R a projective scheme of relative dimension 2 over a dvr R, with algebraically closed residue field and perfect fraction field. Let (X,Δ) be a dlt pair with X $\mathbb{Q}$-factorial and projective and Δ a $\mathbb{Q}$-divisor. If Φ is a $\mathbb{Q}$-divisor such that*

$$0 \leq \Delta - \Phi \leq N_\sigma(X, K_X+\Delta),$$

then (X,Φ) has a good minimal model implies (X,Δ) has a good minimal model.

Proof. Let $f : X \to Y$ the result of running a $(K_X+\Phi)$ minimal model program and let $Y \to Z$ be the ample model of $K_X+\Phi$. Let $\Delta_t = t\Delta + (1-t)\Phi$ so that if $0 < t \ll 0$, then f is also a

$(K_X + \Delta_t)$-MMP. If C generates a $K_X + \Delta_t$-negative extremal ray, suppose that $(K_X + \Phi).C > 0$, thus $-4 \leq (K_X + \Delta).C < 0$ (the left hand by the cone theorem for a 2-dimensional variety over a perfect field in [Tan] applied to X_k). Then following the inequalities in [HMX2014, 5.1], gives a contradiction if $0 < t < \frac{1}{5}$. Thus every step of the $K_X + \Delta_t$ with scaling of an ample divisor is $(K_X + \Phi)$-trivial, so after finitely many steps there is a model $g : X \to W$ contracting the components of $N_\sigma(X, K_X + \Delta_t)$ (this holds with no changes from [HMX2014, 2.7.1]). Thus, as $supp\,(\Delta - \Phi) \subset supp\ N_\sigma(X, K_X + \Delta) = supp\ N_\sigma(X, K_X + \Delta_t)$, then $g_*(K_Y + \Phi)$ is semiample, and [HMX2014, 2.7.2] (which holds using only the resolution) implies that g is a (good) minimal model of (X, Δ). □

14. Existence of Minimal Models Special Case

Lemma 28. [HMX2010, 3.1]*Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k). Let $f : X \to Y$ be a step of the $(K_X + \Delta)$-MMP. If f is birational and V is a non-canonical centre of (X, Δ), then V is not contained in the indeterminacy locus of f, V_0 is not contained in the indeterminacy locus of f_0, and the induced maps $\phi : V \to W$ and $\phi_k : V_k \to W_k$ are birational, where $W = f(V)$. Let $\Gamma = f_*\Delta$. Then $\mathbb{B}_-(Y_0, K_{Y_0} + \Gamma_0)$ contains no non-canonical centres of (Y_0, Γ_0) (so we may repeat the process), and if V is a non-klt centre or $V = X$, then $\phi : V \to W$ and $\phi_0 : V_0 \to W_0$*

are birational contractions. If f is a Mori fibre space, then f_0 is not birational.

Proof. (As in the cited theorem). Suppose f is birational. Let V be a non-canonical centre of (X, Δ). Let $g : X \to Z$ be the contraction of the extremal ray associated to f (so that $f = g$ unless f is a flip). Let $Q = g(V)$, and let $\psi : V \to Q$, be the induced morphism. As every component of V_k is a non-canonical centre of (X_k, Δ_k), then by hypothesis, components of V_k are not contained in $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$, thus ψ_k is defined at each such component, hence is birational. By upper-semicontinuity of the fibers of ψ, ψ is birational, and thus $\phi : V \to Q \to W$, and ϕ_k are both birational.

Now suppose V is a non-klt centre or $V = X$, the above holds in the first case, as non-klt centres are non-canonical. Comparing the discrepancies of the differents of adjunction for Δ_k and Γ_k as in the cited proof shows that ϕ_k, f_k, and thus ϕ are birational contractions. On the other hand, if f is a Mori fibre space, then, as the dimension of the fibers of $f : X \to Y$ are upper-semicontinuous, f_k is not birational. $\square$

Theorem 29. [HMX2014, 3.2] *Let Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k) and is log smooth Let $f : X \to Y$ be a step of the $(K_X + \Delta)$-MMP. If (X_k, Δ_k) has a good minimal model, then we may run $f : X \to Y$ the*

$(K_X+\Delta)$ minimal model program until (X,Δ) is a minimal model and f_k is a semi-ample model of (X_k,Δ_k).

Proof. (Similar proof to the cited theorem). Let $f: X \to Y$ the $(K_X+\Delta)$-MMP with scaling of an ample divisor A. Let $\Gamma = f_*\Delta$ and $B = f_*A$. By construction, $K_Y + tB + \Gamma$ is nef for a $t > 0$. By Lemma 28, $f_k : X_k \to Y_k$ is a weak log-canonical model of $(X_k, tA_k + \Delta_k)$. If $K_X + \Delta$ is not pseudo-effective, then for $t > 0$, the result of f is a Mori fiber space, and by Lemma 28, Y_k is covered by $K_{Y_k} + tB_k + \Gamma_k$-negative curves, contradicting bigness of $K_{X_k} + tA_k + \Delta_k$. Thus $K_X+\Delta$ is pseudo-effective, and given any $\epsilon > 0$, we may run the MMP until $t < 0$. Now we conclude by Theorem 17. ☐

15. Existence of Minimal Models General Case

Theorem 30. *Let (X,Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X+\Delta$ is $\mathbb{Q}$-Cartier, pseudo-effective, and log smooth over R. Then the minimal model of (X,Δ) exists.*

Proof. I start by repeating the reduction of [HMX2014, 6.1]. If necessary, extend R to be complete with algebraically closed residue field k, so the strata of Δ have irreducible fibers over R. Let $f_0 : Y_0 \to X_0$ be the birational morphism of Lemma 25. Under the log smooth hypothesis, and since the strata of Δ have irreducible fibers, and f_0 blows up strata of Δ_0, extend f_0 to a birational morphism $f : Y \to X/R$ which is a composition of smooth blow ups of strata of Δ. Write

$$K_Y + \Gamma = f^* (K_X + \Delta) + E$$

with $\Gamma \geq 0$ and $E \geq 0$, $f_*\Gamma = \Delta$, and $f_*E = 0$. (Y, Γ) is log smooth and the fibres of components of Γ are irreducible. As X_0 is projective variety of dimension 2, by [Tan] (X_0, Δ_0) has a good minimal model, so (Y_0, Δ_0) has a good minimal model by Lemma 26, and similarly (X, Δ) has a good minimal model iff (Y, Γ) has a good minimal model. Thus it suffices to show that (Y, Γ) has a good minimal model. Now we are in close to the situation of the proof of Theorem 15, but with possibly slightly worse singularities.

Replace (X, Δ) by (Y, Γ) and set $\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma (X_0, K_{X_0} + \Delta_0)$ so that $\mathbb{B}_- (X_0, K_{X_0})$ contains no strata of Θ_0. Let $0 \leq \Theta \leq \Delta$ be the unique divisor such that $\Theta_0 = \Theta|_{X_0}$. Let H relatively ample such that $(X, \Delta + H)$ is log smooth over R, and $kK_X + H$ is big. There is a commutative diagram:

$$\begin{array}{ccc}
\pi_* \mathcal{O}_X (m (K_X + \Theta) + H) & \longrightarrow & \pi_* \mathcal{O}_X (m (K_X + \Delta) + H) \\
\downarrow & & \downarrow \\
H^0 (X_0, \mathcal{O}_{X_0} (m (K_{X_0} + \Theta_0) + H_0)) & \longrightarrow & H^0 (X_0, \mathcal{O}_{X_0} (m (K_{X_0} + \Delta_0) + H_0))
\end{array}$$

with surjective columns by Theorem 15, with the bottom row an isomorphism. Applying Nakayama's lemma as in the proof of Theorem 15 gives an isomorphism on the top row, so that $\Theta \geq \Delta - \Delta \wedge N_\sigma (X, K_X + \Delta)$. Again applying Theorem 15, gives $\Theta = \Delta - \Delta \wedge N_\sigma (X, K_X + \Delta)$. Thus $\Delta - \Theta \leq N_\sigma (X, K_X + \Delta)$, so by Lemma 27, it suffices to find a minimal model for (X, Θ). Replace (X, Δ) by (X, Θ), it suffices to assume that $\mathbb{B}_-(X_0, K_{X_0} + \Delta_0)$ contains no strata of Δ_0.

Let A be an ample divisor, we run the minimal model program with scaling of A. Now the assumptions of Theorem 29 apply, so we know that (X, Φ) has a minimal model, (X', Φ') with $K_{X'} + \Phi'$ semiample on both fibers (by [Tan], [Tan4]). Now for any $\epsilon > 0$, and any geometric valuation Γ on X', $\sigma_\Gamma\left(K_{X'_0} + \Phi'_0\right) = \sigma_\Gamma\left(K_{X'_\eta} + \Phi'_\eta\right) = 0$ so that σ_Γ is identically zero, and thus the centre of Γ is not in $\mathbb{B}(K_{X'} + \Phi')$ for any such Γ. $\square$

16. Abundant Termination With Scaling

Theorem 31. [BCHM, 7.1]*Let $\pi : X \to U$ a projective morphism of normal quasi-projective schemes of dimension at most 3. Let V a finite dimensional affine subspace of $WDiv_{\mathbb{R}}(X)$ defined over the rationals. Fix a general ample $\mathbb{Q}$-divisor A over U. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ a rational polytope such that if $\Delta \in \mathcal{C}$, then $K_X + \Delta$ is kawamata log terminal. Then there are finitely many rational maps $\phi_i : X \to Y_i$ over U $1 \le i \le k$, with the property that if $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$ then there is an index $1 \le i \le k$ such that ϕ_i is a log terminal model of $K_X + \Delta$ over U.*

Proof. As in the source, this holds by termination in the big case (holds by [EgSyz]) and non-vanishing in the big case (see [EGSurfDIOP] 3.12+). $\square$

Theorem 32. *Let $(X,\Delta)/R$ be a two dimensional klt pair projective over a dvr R with a good minimal model over R. Then any $(K_X+\Delta)$-minimal model program over R with scaling of an ample divisor terminates.*

Proof. (c.f. [Lai, 26], similar proof to the "Finiteness of Models" above). Let $\phi : (X,\Delta) \to (X_g,\Delta_g)$ a good minimal model (exists by hypothesis). Let $f : X_g \to Z = Proj\ R\left(K_{X_g}+\Delta_g\right)$ the contraction (exists since it's a "Good" minimal model). Then ϕ contracts the divisorial part of $\mathbb{B}(K_X+\Delta)$. Pick $t_0>0$ such that $(X_g,\Delta_g+t_0\phi_*A) = (X_g,\Delta_g+t_0A_g)$ is klt with A ample on X (note A_g is big, not in general nef). For H ample on X_g, let $\phi : X_g \to X'$ be the minimal model program with scaling, which terminates by Theorem 24 (as $\Delta_g+t_0A_g$ is big), and gives a minimal model of $(X_g,\Delta_g+t_0\Delta_g)$ over Z. For any curve contracted by f, $\left(K_{X_g}+\Delta_g\right).C = 0$, hence $K_{X'}+\psi_*\Delta_g = K_{X'}+\Delta' \equiv_Z 0$. Thus curves contracted by ψ have trivial intersection with $K_{X_g}+\Delta_g$, and intersect negatively with A_g. Thus changing t doesn't affect which curves intersect negatively with $K_{X_g}+\Delta_g+tA_g$, and so X' is a minimal model of (X_g,Δ_g+tA_g) for all $t\in(0,t_0]$.

Now $\Delta'+t_0A'$, $A'=\psi_*A_g$ is big implies by Theorem 21, that there exist only finitely many $K_{X'}+\Delta'+t_0A'$ negative extremal rays in $\overline{NE}(X')$. These are all necessarily just intersecting A' negatively, so decreasing t_0, eventually we get to a point where further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (Now we are on X', not over Z). Again shrinking t_0, suppose that

$\psi \circ \phi$ is discrepancy negative w'r't $(X, \Delta + tA)$ for all t in the half open interval $(0, t_0]$. Note that $\mathbb{B}(K_X + \Delta + t_0 A) \subset \mathbb{B}(K_X + \Delta)$, so ψ contracts only the things not contracted by ϕ (which is no divisors). Thus $\psi \circ \phi$ is discrepancy negative on the closed interval $[0, t_0]$, so X' is a minimal model of $(X, \Delta + tA)$ for all $t \in [0, t_0]$. Thus $\mathbb{B}(K_X + \Delta + tA)$ has the same divisorial components for all $t \in [0, t_0]$. So any two minimal models for different $t \in [0, t_0]$ are isomorphic in codimension one.

The rest holds as in the cited proof, with finiteness of models by Theorem 31.

□

17. Running the Minimal Model Program

Theorem 33. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, pseudo-effective, and log smooth over R. Then the minimal model model program with scaling of an ample divisor can be run, terminating in a good minimal model for (X, Δ).*

Proof. This follows from Theorems 30 and 32. □

References

[A1] Artin, M. "Algebraization of formal moduli. I, Global Analysis (Papers in Honor of K. Kodaira), 21–71." (1969).

[A2] Artin, Michael. "Algebraization of formal moduli: II. Existence of modifications." Annals of Mathematics (1970): 88-135.

[BC] Baker, Matthew., and Janos A. Csirik. "On the Isomorphism Between the Dualizing Sheaf and the Canonical Sheaf." (1996).

[BCHM] Birkar, Caucher, et al. "Existence of minimal models for varieties of log general type." Journal of the American Mathematical Society 23.2 (2009): 405.

[Bir] Birkar, Caucher. "Existence of flips and minimal models for 3-folds in char p." arXiv preprint arXiv:1311.3098 (2013).

[Bo] Bosch, Siegfried, et al. "Néron models." (1990).

[BP] Berndtsson, Bo, and Mihai Paun. "Quantitative extensions of pluricanonical forms and closed positive currents." Nagoya Math. J 205 (2012): 25-65.

[CL] Corti, Alessio, and Vladimir Lazić. "New outlook on the minimal model program, II." Mathematische Annalen 356.2 (2013): 617-633.

[CP2009] Cossart, Vincent, and Olivier Piltant. "Resolution of singularities of threefolds in positive characteristic II." Journal of Algebra 321.7 (2009): 1836-1976.

[CP2014] Cossart, Vincent, and Olivier Piltant. "Resolution of Singularities of Arithmetical Threefolds II." arXiv preprint arXiv:1412.0868 (2014).

[CR] Chatzistamatiou, Andre, and Kay Rülling. "Higher direct images of the structure sheaf in positive characteristic." Algebra & Number Theory 5.6 (2012): 693-775.

[CZ] Cascini, Paolo, and DEQI ZHANG. "Effective finite generation for adjoint rings." arXiv preprint arXiv:1203.5204 (2012).

[Das] Das, Omprokash. "On Strongly $ F $-Regular Inversion of Adjunction." arXiv preprint arXiv:1310.8252 (2013).

[Deb] Dèbes, Pierre, et al., eds. Arithmetic and geometry around Galois theory. Springer Science & Business Media, 2012.

[DF] Di Cerbo, Gabriele, and Andrea Fanelli. "Effective Matsusaka's Theorem for surfaces in characteristic p." arXiv preprint arXiv:1501.07299 (2015).

[DI] Deligne, Pierre, and Luc Illusie. "Relèvements modulop 2 et décomposition du complexe de de Rham." Inventiones Mathematicae 89.2 (1987): 247-270.

[EGSurfDIOP] Egbert, Andrew. SurfDIOP. https://divisibility.wordpress.com/2015/06/12/mixed-characteristic-minimal-models-and-invariance-of-plurigenera-for-relative-dimension-2/

[EgSyz] Egbert, Andrew. https://divisibility.wordpress.com/2015/08/07/adjoint-rings-and-syzygies-in-mixed-or-positive-characteristic/

[ELM] Ein, Lawrence, et al. "Asymptotic invariants of base loci." Annales de l'institut Fourier. Vol. 56. No. 6. 2006.

[Ek] Ekedahl, Torsten. "Canonical models of surfaces of general type in positive characteristic." Publications Mathématiques de l'IHÉS 67.1 (1988): 97-144.

[EV] Viehweg, Eckart. Lectures on vanishing theorems. Vol. 20. Springer Science & Business Media, 1992. (The online lecture notes version).

[GH] Griffiths, Phillip, and Joseph Harris. Principles of algebraic geometry. John Wiley & Sons, 2014.

[Gra] Grauert, Hans, Thomas Peternell, and Reinhold Remmert. Several complex variables VII: sheaf-theoretical methods in complex analysis. Vol. 7. Springer Science & Business Media, 1994.

[Gr] Grothendieck, A. "Éléments de géométrie algébrique." New York (1967).

[GW] Görtz, Ulrich, and Torsten Wedhorn. Algebraic Geometry. Vieweg+ Teubner, 2010.

[HK] Hacon, Christopher D., and Sándor Kovács. Classification of higher dimensional algebraic varieties. Vol. 41. Springer Science & Business Media, 2011.

[Hara98] Hara, Nobuo. "Classification of two-dimensional F-regular and F-pure singularities." Advances in Mathematics 133.1 (1998): 33-53.

[Har] Hartshorne, Robin. Algebraic geometry. Vol. 52. Springer Science & Business Media, 1977.

[HS] Hindry, Marc, and Joseph H. Silverman. Diophantine geometry: an introduction. Vol. 201. Springer Science & Business Media, 2000.

[HMX2010] Hacon, Christopher, James McKernan, and Chenyang Xu. "On the birational automorphisms of varieties of general type." arXiv preprint arXiv:1011.1464 (2010).

[HMX2014] Hacon, CHRISTOPHER D., James McKernan, and Chenyang Xu. "Boundedness of moduli of varieties of general type." arXiv preprint arXiv:1412.1186 (2014).

[HX] Hacon, Christopher D., and Chenyang Xu. "Existence of log canonical closures." Inventiones mathematicae 192.1 (2013): 161-195.

[KU] Katsura, Toshiyuki, and Kenji Ueno. "On elliptic surfaces in characteristicp." Mathematische Annalen 272.3 (1985): 291-330.

[KK94] Kollár, János, and Sándor Kovács. "Birational geometry of log surfaces." preprint (1994).

[KM] Kollár, János, and Shigefumi Mori. Birational geometry of algebraic varieties. Vol. 134. Cambridge University Press, 2008.

[Ko] Kollár, János. Singularities of the minimal model program. Vol. 200. Cambridge University Press, 2013.

[Ko2] Kollár, János. Shafarevich maps and automorphic forms. Princeton University Press, 2014.

[Ko1991] Kollár, János, ed. Flips and abundance for algebraic threefolds: a summer seminar at the University of Utah, Salt Lake City, 1991. Société mathématique de France, 1992.

[Lai] Lai, Ching-Jui. "Varieties fibered by good minimal models." Mathematische Annalen 350.3 (2011): 533-547.

[Laz] Lazarsfeld, Robert K. Positivity in algebraic geometry. Springer Science & Business Media, 2004.

[Lev] Levine, Marc. "Pluri-canonical divisors on Kähler manifolds." Inventiones mathematicae 74.2 (1983): 293-303.

[Lev-2] Levine, Marc. "Pluri-canonical divisors on Kähler manifolds, II." Duke Math. J 52.1 (1985): 61-65.

[Lie] Liedtke, Christian. "Algebraic surfaces in positive characteristic." Birational geometry, rational curves, and arithmetic. Springer New York, 2013. 229-292.

[Liu] Liu, Qing, and Reinie Erne. Algebraic geometry and arithmetic curves. Oxford university press, 2002.

[LS] Liedtke, Christian, and Matthew Satriano. "On the birational nature of lifting." Advances in Mathematics 254 (2014): 118-137.

[Ma] Maddock, Zachary. "A bound on embedding dimensions of geometric generic fibers." arXiv preprint arXiv:1407.2529 (2014).

[Mum] Mumford, David, Chidambaran Padmanabhan Ramanujam, and Jurij Ivanovič Manin. Abelian varieties. Vol. 48. Oxford: Oxford university press, 1970.

[Oss] Osserman, Brian. "Notes on Cohomology and Base Change." https://www.math.ucdavis.edu/~osserman/math/cohom-base-change.pdf

[Sat] Satriano, Matthew. "De Rham theory for tame stacks and schemes with linearly reductive singularities." arXiv preprint arXiv:0911.2056 (2009).

[Serre] Serre, Jean-Pierre. Local fields. Vol. 67. Springer Science & Business Media, 2013.

[Siu] Siu, Yum-Tong. "Invariance of plurigenera." arXiv preprint alg-geom/9712016 (1997).

[Siu2] Siu, Yum-Tong. "Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type." Complex geometry. Springer Berlin Heidelberg, 2002. 223-277.

[Suh] Suh, Junecue. "Plurigenera of general type surfaces in mixed characteristic." Compositio Mathematica 144.05 (2008): 1214-1226.

[Tan] Tanaka, Hiromu. "Minimal models and abundance for positive characteristic log surfaces." Nagoya Mathematical Journal (2015).

[Tan2] Tanaka, Hiromu. "The X-method for klt surfaces in positive characteristic." arXiv preprint arXiv:1202.2497 (2012).

[Tan3] Tanaka, Hiromu. "The trace map of Frobenius and extending sections for threefolds." arXiv preprint arXiv:1302.3134 (2013).

[Tan4] Tanaka, Hiromu. "Minimal model theory for surfaces over an imperfect field." arXiv preprint arXiv:1502.01383 (2015).

[Ter] Terakawa, Hiroyuki. "The d-very ampleness on a projective surface in positive characteristic." Pacific J. of Math 187 (1999): 187-198.

[Wal] Waldron, Joe. "Finite generation of the log canonical ring for 3-folds in char $ p$." arXiv preprint arXiv:1503.03831 (2015).

[Xu] Xu, Chenyang. "On base point free theorem of threefolds in positive characteristic." arXiv preprint arXiv:1311.3819 (2013).