MINIMAL MODELS FOR ARITHMETIC THREEFOLD GERMS

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ABSTRACT. In this note, I attempt to finish the log minimal model program proofs started in the SurfDIOP and Adjoint rings paper (see divisibility.wordpress.com). In SurfDIOP, finite generation of the canonical ring for klt, log smooth pairs on a scheme having two dimensional fibers over a DVR. In the syzygies and Adjoint rings note, the minimal model program for such (general type) pairs is studied (and the case that serre duality holds for X). In this, I relax the assumption on kodaira dimension. Sometime in the next week, I'll likely try to combine all three of these in a sane (in terms of organization) manner. Most of this paper is just using results from characteristic 0, checking quickly if they hold over a mixed characteristic DVR, and then applying the finite generation in the big case (from [EGSurfDIOP]).

1. FINITENESS OF MODELS FOR SURFACES

Theorem 1. *Let X be a two dimensional normal variety over a perfect field of characteristic $p > 0$. Let (X, Δ) be a KLT pair and A an ample $\mathbb{Q}$-divisor on X. Then there exists an $\epsilon > 0$, such that the minimal models (and the output of the minimal model program with scaling) for $(X, \Delta + tA)$ are all isomorphic for $t \in [0, \epsilon]$.*

Proof. (Essentially same as [Lai, 26,27] - I'm not sure if this is the correct cite for Lai's thesis which is what I am intending to cite, as it appears to be a springerlink thingy - in any case the proof is essentially the same or simplified, but restated for the slightly different setting, and simplifies due to the fact that birational isomorphism in codimension 1 on a surface gives an isomorphism There is also a proof of essentially the same result in [HMX2014]). Let $\phi : (X, \Delta) \to (X_g, \Delta_g)$ a good minimal model (exists by [Tan]). Let $f : X_g \to Z = Proj\ R\left(K_{X_g} + \Delta_g\right)$ the contraction (exists by since $K_{X_g} + \Delta_g$ is neff, hence semiample by [Tan]). Then ϕ contracts the divisorial part of $\mathbb{B}\left(K_X + \Delta\right)$. Pick $t_0 > 0$ such that $(X_g, \Delta_g + t_0\phi_* A) = (X_g, \Delta_g + t_0 A_g)$ is klt with A ample on X (note A_g is big, not in general nef). For H ample on X_g, let $\phi : X_g \to X'$ be the minimal model program with scaling, which terminates again by [Tan], and gives a minimal model of $(X_g, \Delta_g + t_0\Delta_g)$ over Z. For any curve contracted by f, $\left(K_{X_g} + \Delta_g\right).C = 0$, hence $K_{X'} + \psi_*\Delta_g = K_{X'} + \Delta' \equiv_Z 0$. Thus curves contracted by ψ have trivial intersection with $K_{X_g} + \Delta_g$, and intersect negatively with A_g. Thus changing t doesn't affect which curves intersect negatively with $K_{X_g} + \Delta_g + tA_g$, and so X' is a minimal model of $(X_g, \Delta_g + tA_g)$ for all $t \in (0, t_0]$.

Now $\Delta' + t_0 A'$, $A' = \psi_* A_g$ is big implies by the cone theorem (again found in [Tan]), there exist only finitely many $K_{X'} + \Delta' + t_0 A'$ negative extremal rays in $\overline{NE}\left(X'\right)$. These are all necessarily just intersecting A' negatively, so decreasing t_0, eventually we get to a point where further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (Now we are on X', not over Z). Again shrinking t_0, suppose that $\psi \circ \phi$ is discrepancy negative w'r't $(X, \Delta + tA)$ for all t in the half open interval $(0, t_0]$. Note that $\mathbb{B}\left(K_X + \Delta + t_0 A\right) \subset$

$\mathbb{B}(K_X + \Delta)$, so ψ contracts only the things not contracted by ϕ (which is no divisors). Thus $\psi \circ \phi$ is discrepancy negative on the closed interval $[0, t_0]$, so X' is a minimal model of $(X, \Delta + tA)$ for all $t \in [0, t_0]$. Thus $\mathbb{B}(K_X + \Delta + tA)$ has hte same divisorial components for all $t \in [0, t_0]$. So any two minimal models for different $t \in [0, t_0]$ are isomorphic in codimension one, and as the total dimension is two, they are thus isomorphic. □

2. Cone, Rationality, and contraction / R

(See [EGSurfDIOP] 3.12+)

3. Big Abundance / R

(See [EGSurfDIOP, 3.12+])

4. Contractions given by Extremal Rays / R

(See [EGSurfDIOP, 3.12+])

5. Abundant Termination With Scaling

Theorem 2. [BCHM, 7.1]*Let $\pi : X \to U$ a projective morphism of normal quasi-projective schemes of dimension at most 3. Let V a finite dimensional affine subspace of $WDiv_{\mathbb{R}}(X)$ defined over the rationals. Fix a general ample $\mathbb{Q}$-divisor A over U. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ a rational polytope such that if $\Delta \in \mathcal{C}$, then $K_X + \Delta$ is kawamata log terminal. Then there are finitely many rational maps $\phi_i : X \to Y_i$ over U $1 \leq i \leq k$, with the property that if $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$ then there is an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model of $K_X + \Delta$ over U.*

Proof. As in the source, this holds by termination in the big case (holds by [EgSyz]) and non-vanishing in the big case (see [EGSurfDIOP] 3.12+). ☐

Theorem 3. *Let $(X,\Delta)/R$ be a two dimensional klt pair projective over a dvr R with a good minimal model over R. Then any $(K_X+\Delta)$-minimal model program over R with scaling of an ample divisor terminates.*

Proof. (c.f. [Lai, 26], similar proof to the "Finiteness of Models" above). Let $\phi:(X,\Delta)\to(X_g,\Delta_g)$ a good minimal model (exists by hypothesis). Let $f:X_g\to Z=Proj\ R\left(K_{X_g}+\Delta_g\right)$ the contraction (exists since it's a "Good" minimal model). Then ϕ contracts the divisorial part of $\mathbb{B}(K_X+\Delta)$. Pick $t_0>0$ such that $(X_g,\Delta_g+t_0\phi_*A)=(X_g,\Delta_g+t_0A_g)$ is klt with A ample on X (note A_g is big, not in general nef). For H ample on X_g, let $\phi:X_g\to X'$ be the minimal model program with scaling, which terminates by [EgSyz] (as $\Delta_g+t_0A_g$ is big), and gives a minimal model of $(X_g,\Delta_g+t_0\Delta_g)$ over Z. For any curve contracted by f, $\left(K_{X_g}+\Delta_g\right).C=0$, hence $K_{X'}+\psi_*\Delta_g=K_{X'}+\Delta'\equiv_Z 0$. Thus curves contracted by ψ have trivial intersection with $K_{X_g}+\Delta_g$, and intersect negatively with A_g. Thus changing t doesn't affect which curves intersect negatively with $K_{X_g}+\Delta_g+tA_g$, and so X' is a minimal model of (X_g,Δ_g+tA_g) for all $t\in(0,t_0]$.

Now $\Delta'+t_0A'$, $A'=\psi_*A_g$ is big implies by the cone theorem ([EGSurfDIOP] 3.12+), there exist only finitely many $K_{X'}+\Delta'+t_0A'$ negative extremal rays in $\overline{NE}(X')$. These are all necessarily just intersecting A' negatively, so decreasing t_0, eventually we get to a point where further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (Now we are on X', not over Z). Again shrinking t_0, suppose that $\psi \circ \phi$ is discrepancy negative w'r't $(X, \Delta + tA)$ for all t in the half open interval $(0, t_0]$. Note that $\mathbb{B}\left(K_X + \Delta + t_0A\right) \subset \mathbb{B}\left(K_X + \Delta\right)$, so ψ contracts only the things not contracted by ϕ (which is no divisors). Thus $\psi \circ \phi$ is discrepancy negative on the closed interval $[0, t_0]$, so X' is a minimal model of $(X, \Delta + tA)$ for all $t \in [0, t_0]$. Thus $\mathbb{B}\left(K_X + \Delta + tA\right)$ has the same divisorial components for all $t \in [0, t_0]$. So any two minimal models for different $t \in [0, t_0]$ are isomorphic in codimension one.

The rest holds as in the cited proof, with finiteness of models by Theorem 2.

□

6. Reductions

Theorem 4. [CP2014, 1.1] *Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*

(i) X' is everywhere regular

(ii) π induces an isomorphism $\pi^{-1}\left(Reg\ X\right) \approx Reg\ X$

(iii) $\pi^{-1}\left(Sing\ X\right)$ is a strict normal crossings divisor on X'.

Lemma 5. [HMX2014, 2.8.3] *Let (X, Δ)be a log smooth pair which is a scheme of dimension at most 3 (or wherever resolution holds) where* *the coefficients of* Δ *belong to* $(0,1]$ *and* X *is projective. If* (X,Δ) *has a weak log canonical model, then there is a sequence* $\pi : Y \to X$ *of smooth blow ups of the strata of* Δ *such that if we write* $K_Y + \Gamma = \pi^*(K_X + \Delta) + E$*, where* $\Gamma \geq 0$ *and* $E \geq 0$ *have no common components,* $\pi_*\Gamma = \Delta$ *and* $\pi_* E = 0$ *and if we write*

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma (Y, K_Y + \Gamma),$$

then $\mathbb{B}_- (Y, K_Y + \Gamma')$ *contains no strata of* Γ'*. If* Δ *is a* $\mathbb{Q}$*-divisor, then* Γ' *is a* $\mathbb{Q}$*-divisor.*

Proof. (same as source which is stated over $\mathbb{C}$, since just using resolution). □

Lemma 6. [HX, 2.10] *Let* X/R *a projective scheme of relative dimension* 2 *over a dvr* R*, with algebraically closed residue field and perfect fraction field. Let*(X,Δ) *a dlt pair and* $\mu : X' \to X$ *a proper birational morphism. Write* $K_{X'} + \Delta' = \mu^*(K_X + \Delta) + F$ *where* Δ' *and* F *are effective with no common components. Then* (X,Δ) *has a good minimal model over* U *if and only if* (X',Δ') *has a good minimal model over* U*.*

Proof. Same as in the source, noting that resolution holds in this dimension, and the termination for the big case holds by [EgSyz], and termination in the abundant case holds by Theorem 3. □

Lemma 7. [HMX2014, 5.3]*Let* X/R *a projective scheme of relative dimension* 2 *over a dvr* R*, with algebraically closed residue field and* *perfect fraction field. Let (X, Δ) be a dlt pair with X $\mathbb{Q}$-factorial and projective and Δ a $\mathbb{Q}$-divisor. If Φ is a $\mathbb{Q}$-divisor such that*

$$0 \leq \Delta - \Phi \leq N_\sigma (X, K_X + \Delta),$$

then (X, Φ) has a good minimal model implies (X, Δ) has a good minimal model.

Proof. Let $f : X \to Y$ the result of running a $(K_X + \Phi)$ minimal model program and let $Y \to Z$ be the ample model of $K_X + \Phi$. Let $\Delta_t = t\Delta + (1-t)\Phi$ so that if $0 < t \ll 0$, then f is also a $(K_X + \Delta_t)$-MMP. If C generates a $K_X + \Delta_t$-negative extremal ray, suppose that $(K_X + \Phi).C > 0$, thus $-4 \leq (K_X + \Delta).C < 0$ (the left hand by the cone theorem for a 2-dimensional variety over a perfect field in [Tan] applied to X_k). Then following the inequalities in [HMX2014, 5.1], gives a contradiction if $0 < t < \frac{1}{5}$. Thus every step of the $K_X + \Delta_t$ with scaling of an ample divisor is $(K_X + \Phi)$-trivial, so after finitely many steps there is a model $g : X \to W$ contracting the components of $N_\sigma (X, K_X + \Delta_t)$ (this holds with no changes from [HMX2014, 2.7.1]). Thus, as $supp\,(\Delta - \Phi) \subset supp\ N_\sigma (X, K_X + \Delta) = supp\ N_\sigma (X, K_X + \Delta_t)$, then $g_* (K_Y + \Phi)$ is semiample, and [HMX2014, 2.7.2] (which holds using only the resolution) implies that g is a (good) minimal model of (X, Δ). □

7. Main Theorem

Theorem 8. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect* *fraction field K. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, pseudo-effective, and log smooth over R. Then the minimal model of (X, Δ) exists.*

Proof. I start by repeating the reduction of [HMX2014, 6.1]. If necessary, extend R to be complete with algebraically closed residue field k, so the strata of Δ have irreducible fibers over R. Let $f_0 : Y_0 \to X_0$ be the birational morphism of Lemma 5. Under the log smooth hypothesis, and since the strata of Δ have irreducible fibers, and f_0 blows up strata of Δ_0, extend f_0 to a birational morphism $f : Y \to X/R$ which is a composition of smooth blow ups of strata of Δ. Write

$$K_Y + \Gamma = f^* (K_X + \Delta) + E$$

with $\Gamma \geq 0$ and $E \geq 0$, $f_* \Gamma = \Delta$, and $f_* E = 0$. (Y, Γ) is log smooth and the fibres of components of Γ are irreducible. As (X_0, Δ_0) is projective variety of dimension 2, by [Tan] it has a good minimal model, so (Y_0, Δ_0) has a good minimal model by Lemma 6, and similarly (X, Δ) has a good minimal model iff (Y, Γ) has a good minimal model. Thus it suffices to show that (Y, Γ) has a good minimal model. Now we are in close to the situation of the main invariance of plurigenera theorem of [EGSurfDIOP], but with possibly slightly worse singularities.

Replace (X, Δ) by (Y, Γ) and set $\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma (X_0, K_{X_0} + \Delta_0)$ so that $\mathbb{B}_- (X_0, K_{X_0})$ contains no strata of Θ_0. Let $0 \leq \Theta \leq \Delta$ be the unique divisor such that $\Theta_0 = \Theta|_{X_0}$. Let H relatively ample such that $(X, \Delta + H)$ is log smooth over R, and $kK_X + H$ is big. There is a commutative diagram:

$$\begin{array}{ccc} \pi_*\mathcal{O}_X\left(m\left(K_X+\Theta\right)+H\right) & \longrightarrow & \pi_*\mathcal{O}_X\left(m\left(K_X+\Delta\right)+H\right) \\ \downarrow & & \downarrow \\ H^0\left(X_0,\mathcal{O}_{X_0}\left(m\left(K_{X_0}+\Theta_0\right)+H_0\right)\right) & \longrightarrow & H^0\left(X_0,\mathcal{O}_{X_0}\left(m\left(K_{X_0}+\Delta_0\right)+H_0\right)\right) \end{array}$$

with surjective columns by the main invariance of plurigenera theorem of [EGSurfDIOP], with the bottom row an isomorphism. Applying Nakayama's lemma as in [EGSurfDIOP, EgSyz] gives an isomorphism on the top row, so that $\Theta \geq \Delta - \Delta \wedge N_\sigma\left(X, K_X+\Delta\right)$. Again applying the invariance of plurigenera in [EGSurfDIOP], gives $\Theta = \Delta - \Delta \wedge N_\sigma\left(X, K_X+\Delta\right)$. Thus $\Delta - \Theta \leq N_\sigma\left(X, K_X+\Delta\right)$, so by Lemma 7, it suffices to find a minimal model for (X, Θ). Replace (X, Δ) by (X, Θ), it suffices to assume that $\mathbb{B}_-\left(X_0, K_{X_0}+\Delta_0\right)$ contains no strata of Δ_0.

Let A be an ample divisor, we run the minimal model program with scaling of A. Now the assumptions of [HMX2014, 3.1] apply, so we can actually find each step $f : X^i \to Y^i$ of the minimal model program with scaling induces a (WLCM) $f_0 : X_0^i \to Y_0^i$ of $\left(X_0, tA_0+\Delta_0\right)$. Now applying finiteness of models gives the termination[EgSyz]. Now, we know that (X, Φ) has a minimal model, (X', Φ') with $K_{X'}+\Phi'$ semi-ample on both fibers (by [Tan]). Now for any $\epsilon > 0$, and any geometric valuation Γ on X', $\sigma_\Gamma\left(K_{X_0'}+\Phi_0'\right) = \sigma_\Gamma\left(K_{X_\eta'}+\Phi_\eta'\right) = 0$ so that σ_Γ is identically zero, and thus the centre of Γ is not in $\mathbb{B}\left(K_{X'}+\Phi'\right)$ for any such Γ. □

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