LOG MINIMAL MODELS FOR ARITHMETIC THREEFOLD GERMS

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ABSTRACT. In this note, I finish the log minimal model program proofs started in [EGSurfDIOP] and [EgSyz]. In the first note, finite generation of the log canonical ring is proven for Kawamata Log Terminal, log smooth pairs of general type on an algebraic space having two dimensional fibers over a DVR. In the [EgSyz], the minimal model program for such pairs is studied. In this note, I study pairs of any Kodaira dimension.

CONTENTS

1. Intersection Theory on X/R

Suppose X/R is a proper algebraic space of relative dimension 2 over a discrete valuation ring R with residue field k of characteristic $p > 0$ and perfect fraction field K (See for example the compact Shimura Varieties mentioned in [Suh, 0.1]).

Lemma 1. [KU, 9.3] *Let $\varphi : X \to Spec(R)$ be an algebraic two-dimensional space, smooth, proper, separated, and of finite type over $spec(R)$, where R is a DVR with algebraically closed residue field k, and field of fractions K. Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ. If D, D' are divisors on X, then*

$$K_{X_k}^2 = K_{X_K}^2$$
$$(K_{X_k} \cdot D_k)_{X_k} = (K_{X_K} \cdot D_K)_{X_K}$$
$$(D_k \cdot D'_k)_{X_k} = (D_K \cdot D'_K)_{X_K}.$$

If X/R is smooth, then Lemma 1 applies to show that for any divisors C, D extending to both fibers, we can define $C.D = (C|_{X_k}.D|_{X_k})_{X_k}$. On the other hand, if X/R is merely normal and proper, but is actually a scheme, the resolution of singularities, Theorem 3, holds. Thus the intersection theory can be defined as in [Tan, Def 3.1]: $f : X' \to X$ is a resolution, and $C.D = f^*C.f^*D$ for two divisors C, D on Y/R. By properness, this intersection extends by linearity to Weil divisors with

$\mathbb{Q}$ or $\mathbb{R}$ coefficients. Numerical equivalence and $N^1(X)_{\mathbb{Q},\mathbb{R}}$ are then defined as usual.

Note the two different hypothesis here: The invariance of plurigenera theorem requires a proper, smooth algebraic space of relative dimension 2, while the existence of minimal models requires the intermediate steps of the minimal model program to have some intersection theory, and thus be schemes. Recall that a Cartier divisor C on a variety is usually called nef if $C.D \geq 0$ for all curves C.

Lemma 2. *Let S be an affine Dedekind scheme and $f : X \to S$ a projective morphism. Let $\mathcal{L}$ be an invertible sheaf on X such that $\mathcal{L}_s$ is nef for every closed point $s \in S$. Then $\mathcal{L}$ is nef.*

Proof. By [Liu, 5.3.24], the statement holds when nef is replaced by ample. Now, c.f. [Laz, 1.4.10], it suffices to choose an ample divisor H which restricts to X_s, so that $D_s + \epsilon H_s$ is ample for all sufficiently small ϵ. Then $D + \epsilon H$ is ample for all sufficiently small ϵ, and so D is nef. □

2. Log Resolution

The following statement of resolution will be used. So for example, if X is an algebraic variety over a number field K then, by [MP, III.2], there is a completed arithmetical variety $\overline{X}$ of dimension $dim(X)+1$:

$$\overline{X} \to \overline{Spec\ \mathcal{O}_K}.$$

(Note that a complete DVR is excellent).

Theorem 3. [CP2014, 1.1] *Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*

(i) X' is everywhere regular

(ii) π induces an isomorphism $\pi^{-1}(Reg\ X) \approx Reg\ X$

(iii) $\pi^{-1}(Sing\ X)$ is a strict normal crossings divisor on X'.

The following Statement of Log Resolution will be used in the Log Smooth case (i.e. the case with (X, Δ) a pair with X smooth and Δ simple normal crossings.)

Theorem 4. [CP2014, 4.3] *Let S be a regular Noetherian irreducible scheme of dimension three which is excellent and $\mathcal{I} \subset \mathcal{O}_S$ be a nonzero ideal sheaf. There exists a finite sequence*

$$S =: S(0) \leftarrow S(1) \leftarrow \cdots \leftarrow S(r)$$

with the following properties:

(i) for each j, $0 \leq j \leq r-1$, $S(j+1)$ is the blowing up along a regular integral subscheme $\mathcal{Y}(j) \subset \mathcal{S}(j)$ with

$$\mathcal{Y}(j) \subseteq \left\{ s_j \in \mathcal{S}(j) : \mathcal{I}\mathcal{O}_{S(j),s_j} \text{ is not locally principal} \right\}.$$

(ii) $\mathcal{I}\mathcal{O}_{S(r)}$ is locally principal.

3. Effective KLT Kawamata Viehweg Vanishing

In this section, I note that a modification of Tanaka's vanishing theorem [Tan2] gives a version of the (log) Kawamata Viehweg Vanishing in which the multiple of the nef divisor is at least computable. There is a similar recent result in [DF], which is more restrictive (although with a much better constant) in that it requires a smooth variety rather than a normal projective klt pair. First, a theorem of Terakawa:

Theorem 5. [Ter] *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X. Assume that either*

(1) $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or

(2) X is of general type with

$p \geq 3$ and $(D^2) > vol(X)$ or

$p = 2$ and $(D^2) > max\{vol(X), vol(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all $i > 0$.

Recall the following covering Lemma:

Lemma 6. *Let X be an n-dimensional smooth variety. Let D be a $\mathbb{Q}$-divisor such that the support of the fractional part $\{D\}$ is simple normal crossing. Moreover, suppose that, for the prime decomposition $\{D\} = \sum_{i\in I} \frac{b^{(i)}}{a^{(i)}} D^{(i)}$, no integers $a^{(i)}$ are divisible by p. Then there exists a finite surjective morphism $\gamma : Y \to X$ from a smooth variety Y with the following properties:*

(1) The field extension $K(Y)/K(X)$ is a Galois extension.

(2) $\gamma^* D$ is a $\mathbb{Z}$-divisor (it seems the degree of the extension is an m clearing the denominators of the $D^{(i)}$.)

(3) $\mathcal{O}_X\left(K_X+\ulcorner D\urcorner\right)\approx\left(\gamma_*\mathcal{O}_Y\left(K_Y+\gamma^*D\right)\right)^G$, *where* G *is the Galois group of* $K(Y)/K(X)$.

(4) If D' *is a* $\mathbb{Q}$*-divisor such that* $\{D'\}=\{D\}$, *then* γ^*D' *is a* $\mathbb{Z}$*-divisor, and* $\mathcal{O}_X\left(K_X+\ulcorner D'\urcorner\right)\approx\left(\gamma_*\mathcal{O}_Y\left(K_Y+\gamma^*D'\right)\right)^G$.

Remark 7. Several aspects of the proof of the above Lemma will be used in the following.

Applying the above to Tanaka's proof seems to give the following:

Theorem 8. *(*[Tan2, 2.6] *Weak Effective Kawamata-Viehweg Vanishing Theorem) Let* X *be a smooth projective surface over an algebraically closed field of characteristic* $p>2$. *Let* N *be a nef and big* $\mathbb{R}$*-cartier and* B *a nef and big* $\mathbb{Q}$*-divisor whose fractional part is simple normal crossing, whose fractional part has no denominators divisible by the characteristic. Then there exists an* r*, computable in terms of the intersection numbers of the components of* B, N*, and* K_X*, such that*

$$H^i\left(X,\mathcal{O}_X\left(K_X+\ulcorner B\urcorner+rN+N'\right)\right)=0$$

for every $i>0$*, and every nef* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* N' *such that* $rN+N'$ *is a* $\mathbb{Z}$*-divisor.*

Proof. (A slightly different proof than the cited Theorem). The fractional part of $B+rN+N'$ is equal to the fraction part of B when $rN+N'$ is a $\mathbb{Z}$-divisor. Thus we apply Lemma 6 to obtain a degree m cover $\gamma:Y\to X$ (independent of r and N') where Y is a smooth surface. Then

$$H^1\left(K_X+\ulcorner B\urcorner+rN+N'\right)$$

$$= H^1 \left(K_X + \ulcorner B + rN + N' \urcorner\right)$$

$$= H^1 \left(\gamma_* \left(K_Y + \gamma^* \left(B + rN + N'\right)\right)\right)^G$$

$$= H^1 \left(Y, K_Y + \gamma^* B + r\gamma^* N + \gamma^* N'\right)^G .$$

the last term vanishing by Terakawa's Theorem and some $r \gg 0$, as by the proof of [KMM, 1-1-1], the cover can be made general type.

Now I claim it is possible to compute an r such that the last term above is zero using the intersections and degrees of components of $B = \sum a_i \Gamma_i$. It will suffice to show that Y is general type and that $(B + rN)^2 > K_Y^2$, so that Theorem 5 can be applied.

As B is big and nef, there exists an ample divisor A and an effective divisor D such that $B \equiv A - \frac{1}{j} D$ for all $j \gg 0$. Following the proof of [KMM, 1-1-1], let $M = kA$ be very ample (we can compute k by [DF]) and such that m clears the denominators of the components of B and such that $mM - \Gamma_i$ is very ample for all i (again we can compute such an m by [DF]: in the case that $\Gamma_i^2 < 0$, then ensure k is large enough so that $kA.\Gamma_i > -\Gamma_i^2$, take $\Gamma_i' = \Gamma_i + (k-1) K_X$ and by [DF, Theorem 1.2] find (computable) m' such that $m'M - \Gamma_i' = m''M - \Gamma_i$ is very ample. Replace m by m'', and letting $H_i \in |mM - \Gamma_i|$ for all i (thus $\frac{1}{m}(\Gamma_i + H_i) \sim M$ is very ample), we then have (again c.f. the proof of [KMM, 1-1-1])

$$K_Y$$

$$= \tau^* K_X + (m-1) \left(\sum (\tau^* \Gamma_i)_{red} + \sum \left(\tau^* H_k^{(i)} \right)_{red} \right)$$

$$= \tau^* K_X + \tau^* \left(\sum_i \left(\Gamma_i + H_i - \frac{1}{m} (\Gamma_i + H_i) \right) \right)$$

$$= \tau^* K_X + \tau^* ((m-1) M)$$

so that Y is general type, and $vol(Y) = [((m+1)k+1)A]^2$, so it suffices to pick r large enough that $(B+rN)^2 > [((m+1)k+1)A]^2$ or, taking $j \gg 0$, such that $(B+rN) > [((m+1)k+1)B]^2$. □

Now I restate Tanaka's vanishing theorem, with a note on the computability in certain circumstances.

Theorem 9. *(Effective KLT Kawamata-Viehweg Vanishing for normal surfaces c.f.* [Tan2, 2.11]*) Let (X, Δ) be a normal projective klt surface over an algebraically closed field of characteristic $p > 2$, where $K_X + \Delta$ is an effective $\mathbb{R}$-divisor. Let N be a nef and big $\mathbb{R}$-cartier $\mathbb{R}$-divisor. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D - (K_X - \Delta)$ is nef and big. Then there exists an r_0* [1] *such that*

$$H^i (X, \mathcal{O}_X (D + rN + N')) = 0$$

for every $i > 0$, every positive real number $r \geq r_0$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor N' such that $rN + N'$ is a Cartier divisor.

Proof. The computability follows easily from [Tan2, 2.11] except that Weak Kawamata Viehweg Vanishing in the proof is replaced by Theorem 8. Since, in this note I can apply this theorem when (X, Δ) is log

[1]By Theorem 8, in the log-smooth case, r_0 is computable in terms of the number of components Δ and in terms of the self intersections of these components, and their multiplicities. Otherwise Tanaka's original statement gives just the existence of an r_0.

smooth (hence X is smooth) and $\llcorner\Delta\lrcorner = 0$, here is the proof in that simple case:

$$H^1(D + rN + N')$$
$$= H^1(K_X + D - K_X - \llcorner\Delta\lrcorner + rN + N')$$
$$= H^1(K_X + D - K_X + \ulcorner -\Delta \urcorner + rN + N')$$
$$= H^1(K_X + \ulcorner D - K_X - \Delta \urcorner + rN + N')$$

since we can move the integral divisor $D - K_X$ into the round up. This last term vanishes by Theorem 8. The vanishing of H^2 follows easily using Serre Duality. □

4. Invariance of Plurigenera Special Case

Lemma 10. [KU, 9.4] *Let $\varphi : X \to Spec(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $spec(R)$, where R is a DVR with algebraically closed residue field k, and field of fractions K. Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ. If X_k contains an exceptional curve of the first kind e, there exists a DVR $\tilde{R} \supset R$, with residue field isomorphic to k, and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \to \tilde{Y}$ over $Spec\left(\tilde{R}\right)$ such that on the closed fibre, π induces the contraction of the exceptional curve e. Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.*

Proof. (Follows easily from the cited theorem, although I make a small observation about the extension of residue fields). By [A1, Cor 6.2] $Hilb_{X/Spec(R)}$ is represented by an algebraic space $\mathscr{H}$ which is locally of finite type over $Spec\,(R)$. Let Y be the irreducible component containing the point $\{e\}$ corresponding to the exceptional curve e on the special fiber. Then $e \approx \mathbb{P}^1_k$ and $N_{e/X} \approx \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, so Y is regular at $\{e\}$ and of dimension 1. Since e is fixed in the special fiber, the structure morphism $p : Y \to Spec\,(R)$ is surjective. By [Bo, 2.2.14][2], we can find an étale cover $\tilde{R} \supset R$ and a morphism $j : Spec\left(\tilde{R}\right) \to Y$ over $Spec\,(R)$ with $j\,(\tilde{o}) = \{e\}$ (if R is not already complete, then first extend R to a complete DVR using [Gr, 0.6.8.2,3] so that $\tilde{R}$ is again a DVR). As k is assumed algebraically closed, and $\tilde{R} \to R$ is unramified, then the extension of residue fields is finite and separable at the closed point of R, and hence an isomorphism of residue fields. Let $\hat{p} : \mathscr{E} \to Spec\left(\tilde{R}\right)$ be the pull-back of the universal family over Y. As the closed fibre $\mathscr{E}_0$ is a projective line, we may choose the morphism j in such a way that the generic fibre $\mathscr{E}_1$ of $\hat{p}$ is also a projective line. Moreover, $\mathscr{E}$ can be considered as a smooth closed algebraic subspace of codimension 1 in $\hat{X} = X \otimes \tilde{R}$. By Lemma 1, $-1 = \mathscr{E}_0^2 = \mathscr{E}_1^2$. Hence $\mathscr{E}_1$ is also an exceptional curve of the first kind. Hence by [A2, Cor 6.11] there exists a contraction morphism $\pi : \hat{X} \to \hat{X}$ over $Spec\left(\tilde{R}\right)$ which contracts $\mathscr{E}$ to a section of $\tilde{\varphi} : \tilde{X} \to Spec\,(R)$ where $\tilde{\varphi}$ is proper, smooth, separated and of finite type over $Spec\left(\tilde{R}\right)$.

□

[2]If k is perfect, the proof of the cited lemma goes seems to go through when $f : X \to S$ is merely smooth at $x \in X$ and $f : X \to S$ is locally of finite type.

Lemma 11. [KK94, 2.3.5] *Let (S, B) be a log-canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C.(K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

Theorem 12. [Tan, 4.4, 4.6] *Let X be a projective normal surface and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let H be an $\mathbb{R}$-cartier ample $\mathbb{R}$-divisor. Then the following assertions hold:*

(1) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$

(2) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i]$

(3) Each C_i in (1) and (2) is rational or $C_i = B_j$ for some B_j with $B_j^2 < 0$.

(4) Each C_i in (1) and (2) satisfies $0 < -C_i.(K_X + \Delta) \leq 3$.

Proposition 13. *Let (X, Δ) be a terminal log smooth pair of relative dimension 2 over a DVR R with algebraically closed residue field k of characteristic $p > 2$ and perfect fraction field K. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\kappa(K_{X_k} + \Delta_k) = 2$. Assume that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$. Then there exists an m_0 such that for $m > m_0$,*

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

The proof of proposition 13 is given in the following claims.

Claim. Assumptions as above, after passing to an extension R' of R, there is a proper, smooth algebraic space X^{min}/R' and an R' morphism

$X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef.

Proof. As k is algebraically closed of characteristic $p > 0$, then by the Cone Theorem 12,

$$\overline{NE}\left(X_k\right) = \overline{NE}\left(X_k\right)_{K_{X_k}+\Delta_k \geq 0} + \sum \mathbb{R}_{\geq 0}\left[C_i\right]$$

with each C_i is rational or $C_i = B_j$ for some B_j a component of Δ with $B_j^2 < 0$. Under the assumption that $\mathbb{B}_-\left(K_{X_k} + \Delta_k\right) \wedge \Delta_k = \emptyset$, then actually each C_i is rational and is not a component of Δ_k. Thus $C_i.\Delta_k \geq 0$ so $C_i.K_{X_k} < 0$, and if $C_i^2 > 0$, then by [KK94, 2.3.3], then $-\left(K_{X_k} + \Delta_k\right)$ is ample, which contradicts the assumption that $\kappa\left(K_{X_k} + \Delta_k\right) = 2$. Thus Theorem 11 implies $C_i \approx \mathbb{P}^1$ and C_i can be contracted to a log-canonical point, which is actually a smooth point under the terminal assumption. Thus C_i is an exceptional curve of the first kind, so it is possible to apply Lemma 10.

By Lemma 10, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \to \tilde{X}$ induces the contraction of C_i on X_k and X_K, and further $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since k is algebraically closed, it induces identity on residue fields. Now I need to work with $\tilde{X}$. $\tilde{X}_k$ is projective, so the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no problem in extending R. As the Picard number of the special fiber drops each step, there are only finitely many steps. □

Claim 14. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef.

Proof. It suffices to apply Lemma 2 to X_k^{min}, and then restrict to X_K. □

Claim 15. There is an m_0 such that for $m_0|m$, we have

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Proof. Suppose first that (X_k, Δ_k) is big. By the claims 4 and 14, we achieve X^{min} such that $K_X + \Delta$ is nef on both X_k^{min} and X_K^{min}. By Theorem 9 applied to the special fiber (and the semicontinuity theorem) there exists an $m_0 \gg 0$ such that for $m_0 < m$ and $i > 0$, we have

$$h^i\left(m\left(K_{X_K} + \Delta_K\right)\right) = h^i\left(m\left(K_{X_k} + \Delta_k\right)\right) = 0.$$

Thus, applying the invariance of Euler characteristic [Liu, 5.3.22], and birational invariance of the plurigenera, it follows that for $m > m_0$,

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

□

Remark 16. In fact, if $\Delta = 0$, then, at the end of the above proof, Ekedahl's vanishing Theorem [Ek] can be applied and $m_0 = 2$, in general m_0 can be computed by Theorem 9.

5. Invariance of Plurigenera General Case

Theorem 17. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists an m_0 such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

Proof. Since the hypothesis and conclusion are preserved by base change, then after extending R if necessary, we may assume that k is algebraically closed and that R is complete. To begin, I make the reduction of [HMX2010, 1.6]. Replacing (X, Δ) by a blow-up of strata, assume (X, Δ) is terminal. Recall that the log-canonical ring of $K_{X_k} + \Delta_k$ is finitely generated (c.f. [Tan, 7.1]) so $N_\sigma\left(K_{X_k} + \Delta_k\right)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma\left(K_{X_k} + \Delta_k\right)$$

and let $0 \leq \Theta \leq \Delta$ be a $\mathbb{Q}$-divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. Letting $m \gg 0$ to fit the hypothesis of Proposition 13, and sufficiently divisible such that $m\left(K_{X_k} + \Theta_k\right)$ is integral, then by definition of N_σ,

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_k} + \Theta_k\right)\right).$$

Furthermore, by Proposition 13,

$$h^0\left(m\left(K_{X_k} + \Theta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Theta_K\right)\right).$$

As $\Theta \leq \Delta$, the theorem follows by semicontinuity. $\square$

Remark 18. Applying Lemma 37 to the proof of Theorem 17, the minimal models of (X, Δ) and (X, Θ) are actually the same, furthermore, all the additional curves in Δ which appear after blowing up strata must be -1 curves (since contracting them gives a point of the log-smooth surface X_k). Using these two facts, it is possible to compute m_0 as in Theorem 13.

6. Finite Generation (Big Case)

Corollary 19. *Situation as above, after possibly extending R, the canonical ring $R(K_X + \Delta)$ is finitely generated over R.*

Proof. By [Tan], there is $m \gg 0$, such that

$$\bigoplus_{m \geq 0} H^0 \left(ml \left(K_{X_k} + \Delta_k\right)\right)$$

is generated in degree 1, so that there is a surjection

$$S^l H^0 \left(m \left(K_{X_k} + \Delta_k\right)\right) \twoheadrightarrow H^0 \left(ml \left(K_{X_k} + \Delta_k\right)\right).$$

Consider the following diagram:

$$\begin{array}{ccc} S^l H^0 \left(m \left(K_X + \Delta\right)/R\right) & \xrightarrow{\alpha} & H^0 \left(ml \left(K_X + \Delta\right)/R\right) \\ \downarrow & & \downarrow \\ S^l H^0 \left(m \left(K_{X_k} + \Delta_k\right)\right) & \xrightarrow{\twoheadrightarrow} & H^0 \left(ml \left(K_{X_k} + \Delta_k\right)\right) \end{array}$$

The vertical maps are surjective by the first part of the theorem (and applying Nakayama's lemma and cohomology and base change to pull back the generators of the k-modules). Now applying the version of Nakayama's Lemma given in [GH, Chapter 5.3], surjectivity of the bottom map implies surjectivity of α, and thus $R(K_X+\Delta)$ is finitely generated over R. □

Remark 20. It is possible to compute an upper bound for the degree of generation of the log-canonical using Theorem 9. This is explored in more detail in [EgDeg].

Definition 21. Let $(X, \sum S_i)$ a log smooth, log-canonical projective pair where X is a two dimensional, normal variety over a perfect field of characteristic $p>0$. with S_i distinct prime divisors, $V=\sum \mathbb{R}S_i \subset Div_{\mathbb{R}}(X)$. Define the following sets, the first of which is clearly a polytope.

$$\mathcal{L}(V)=\left\{\sum_{i=1}^{n} a_i S_i \in V \mid a_i \in [0,1]\right\}$$
$$\mathcal{E}(V)=\{\Delta \in \mathcal{L}(V) \mid K_X+\Delta \sim_{\mathbb{R}} D \geq 0\}.$$

Given $f: X \to Y$ a birational contraction, let $\mathcal{C}_f(V)$ denote the closure of $\mathcal{L}(V)$ of

$$\{\Delta \in \mathcal{E}(V) \mid f \text{ is a log terminal model of } (X,\Delta)\}.$$

C.f. [CL2013, 2.13], (the statement is for characteristic 0 in any dimension by [SC, 3.4], but in the surface case here merely relies on [Tan, 0.2]) then there are birational contractions $f_i: X \to Y_i$ such that

$\mathcal{C}_{f_i}(V), ..., \mathcal{C}_{f_k}(V)$ are rational polytopes and

$$\mathcal{E}(V) = \bigcup_{i=1}^{k} \mathcal{C}_{f_i}(V)$$

so that $\mathcal{E}(V)$ is also a polytope.

Proposition 22. [H] *Let X be a two dimensional $\mathbb{Q}$-factorial normal variety over a perfect field of characteristic $p > 0$. Let $\{(X, \Delta_i)\}_{i \in \{1,...,k\}}$ be a big, log smooth, $\mathbb{Q}$-Cartier KLT pair for each i and such that $K_X + \sum \Delta_i$ has simple normal crossings support. Then the ring*

$$R(X, K_X + \Delta_1, ..., K_X + \Delta_k)$$

is finitely generated.

Proof. I sketch a proof based on [CL2013, 2.18] which applies in the KLT case (Hashizume's Theorem seems to give the more general, log-canonical case). Let $\sum supp(\Delta_i) = \sum S_i$ for S_i distinct prime divisors, $D_i = \ell(K_X + \Delta_i + A)$ and $\mathcal{M} = \sum_{i=1}^{p} \mathbb{Z} D_i$. Let $M^{(N)} = \sum_{i=1}^{p} \mathbb{Z} N D_i$. As in the cited theorem, there is a constant M' and generators of $\mathcal{M} \cap \mathbb{R}_+ (K_X + \mathcal{C}_i)$ for all i of the form $M'(K_X + B)$ for some $B \in \mathcal{L}(V)$. Applying abundance ([Tan]), there is an $M \in \mathbb{N}$ such that for all i, j where B_{ij} span each $\mathcal{C}_i$, such that $M(f_i)_* (K_X + B_{ij})$ is basepoint free. Thus, c.f. [CZ, 2.11], we are reduced to proving the following: Let (X, B_i) a klt pair for each i and $L_i = a_i (K_X + B_i)$ has basepoint free linear system for each i, (we can assume here that $K_X + B_i$ is big). Let $G = \sum_{i=1}^{k} b_i L_i$ for positive integers b_i, and assume that $b_\ell > n + 1$.

Then the natural map

$$H^0(X, \mathcal{O}_X(G)) \otimes H^0(X, \mathcal{O}_X(L_\ell)) \to H^0(X, \mathcal{O}_X(G + L_\ell))$$

is surjective. The rest follows the proof of [CZ, 2.7], but substituting the Vanishing Theorem 9 when necessary for the usual Kawamata Viehweg Vanishing. □

Corollary 23. *Let $\{(X, \Delta_i)\}_{i\in\{1,...,n\}}$ be a collection of klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta_i$ is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R for each i, and such that $K_X + \sum \Delta_i$ has simple normal crossings support. Then the ring*

$$R(X, K_X + \Delta_1, ..., K_X + \Delta_n)$$

is finitely generated over R.

Proof. C.f. the proof of [CZ, 2.11], it suffices to prove that if $G = \sum_{i=1}^{k} m_i (a(K_X + \Delta_i))$, with fixed $a \in \mathbb{N}$, $m_1, ..., m_k \geq 0$, and $l \in \{1, ..., l\}$ then there exists m_l such that the following maps are surjective

$$H^0(X, \mathcal{O}_X(G - a(K_X + \Delta_i))) \otimes H^0(X, \mathcal{O}_X(a(K_X + \Delta_i)))$$

$$\to H^0(X, \mathcal{O}_X(G)).$$

By applying Nakayama's Lemma and Theorem 17 as in the proof of Theorem 19, it suffices to show the surjectivity on the central fiber which holds by proposition 22. □

7. Finiteness of Models For Surfaces

Theorem 24. *Let X be a two dimensional normal variety over a perfect field of characteristic $p > 0$. Let (X, Δ) be a log smooth KLT pair and A an ample $\mathbb{Q}$-divisor on X with a good minimal model (which exists for non-negative Kodaira dimensions by [Tan]). Then there exists an $\epsilon > 0$ such that the minimal models (and the output of the minimal model program with scaling) for $(X, \Delta + tA)$ are all isomorphic for $t \in [0, \epsilon]$.*

Proof. (Essentially same as [Lai, 26,27] and simplifies due to the fact that a birational isomorphism in codimension 1 on a surface gives an isomorphism). Let $\phi : (X, \Delta) \to (X_g, \Delta_g)$ be a good minimal model. Let $f : X_g \to Z = Proj\ R\left(K_{X_g} + \Delta_g\right)$ the contraction (which exists by since $K_{X_g} + \Delta_g$ is nef, hence semiample by [Tan]). Then ϕ contracts the divisorial part of $\mathbb{B}\left(K_X + \Delta\right)$. Pick $t_0 > 0$ such that $(X_g, \Delta_g + t_0\phi_* A) = (X_g, \Delta_g + t_0 A_g)$ is klt with A ample on X (note A_g is big, not in general nef). For H ample on X_g, let $\phi : X_g \to X'$ be the minimal model program with scaling, which terminates (again by [Tan]), and gives a minimal model of $(X_g, \Delta_g + t_0\Delta_g)$ over Z. For any curve contracted by f, $\left(K_{X_g} + \Delta_g\right).C = 0$, hence $K_{X'} + \psi_*\Delta_g = K_{X'} + \Delta' \equiv_Z 0$. Thus curves contracted by ψ have trivial intersection with $K_{X_g} + \Delta_g$, and intersect negatively with A_g. Thus changing t does not affect which curves intersect negatively with $K_{X_g} + \Delta_g + tA_g$, and so X' is a minimal model of $(X_g, \Delta_g + tA_g)$ for all $t \in (0, t_0]$.

Now $\Delta' + t_0 A'$ with $A' = \psi_* A_g$ (which is big) implies by the Theorem 12, that there exist only finitely many $K_{X'} + \Delta' + t_0 A'$ negative extremal rays in $\overline{NE}\left(X'\right)$. These are all necessarily just intersecting A' negatively, so decreasing t_0, eventually we get to a point where a further decrease in t_0 doesn't change the number of negative extremal rays. Pick such a t_0 (Now we are on X', not over Z). Again shrinking t_0, suppose that $\psi \circ \phi$ is discrepancy negative w'r't $(X, \Delta + tA)$ for all t in the half open interval $(0, t_0]$. Note that $\mathbb{B}\left(K_X + \Delta + t_0 A\right) \subset \mathbb{B}\left(K_X + \Delta\right)$, so ψ contracts only the curves not contracted by ϕ. Thus $\psi \circ \phi$ is discrepancy negative on the closed interval $[0, t_0]$, so X' is a minimal model of $(X, \Delta + tA)$ for all $t \in [0, t_0]$. Thus $\mathbb{B}\left(K_X + \Delta + tA\right)$ has the same divisorial components for all $t \in [0, t_0]$. Hence, any two minimal models for different $t \in [0, t_0]$ are birational and isomorphic in codimension one, and as the total dimension is two, they are thus isomorphic. □

8. Cone, Rationality, and contraction / R

Theorem 25. [CL, 3.5] *Let X be a normal scheme of relative dimension 2, proper over S, the spectrum of a DVR which has perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier divisors on X. Assume that $R\left(X; D_1, ..., D_r\right)$ is finitely generated, and let $D : \mathbb{R}^r \ni (\lambda_1, ..., \lambda_r) \mapsto \sum \lambda_i D_i \in Div_{\mathbb{R}}\left(X\right)$ be the tautological map.*

(1) The support of R is a rational polyhedral cone.

(2) If Supp R contains a big divisor and $D \in \sum \mathbb{R}_+ D_i$ is pseudoeffective, then $D \in$ Supp R.

(3) There is a finite rational polyhedral subdivision Supp $R = \bigcup C_i$ *such that* σ_Γ *is a linear function on* C_i *for every geometric valuation* Γ *of* X*. Furthermore, there is a coarsest subdivision with this property, in the sense that, if* i *and* j *are distinct, there is at least one geometric valuation* Γ *of* X *such that (the linear extensions of)* $\sigma_\Gamma|_{C_i}$ *and* $\sigma_\Gamma|_{C_j}$ *are different.*

(4) There is a finite index subgroup $\mathbb{L} \subset \mathbb{Z}^r$ *such that for all* $n \in \mathbb{N}^r \cap \mathbb{L}$*, if* $D(n) \in$ *Supp* R*, then* $\sigma_\Gamma(D(n)) = mult_\Gamma |D(n)|$*.*

Proof. (1, 2) follow easily from the cited theorem. For $3, 4$ note that [ELM, 4.7] is stated for an arbitrary Noetherian scheme, and this theorem is the only result used in the proof of this result which is given in [CL, 3.5]. □

Theorem 26. *([CL, 3.6]) Let* (X, Δ) *be a klt pair of relative dimension* 2 *over a DVR* R *with perfect residue field* k *of characteristic* $p > 0$ *and perfect fraction field* K*. Assume that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier, and simple normal crossings over* R*. If* $K_X + \Delta$ *is pseudoeffective, then it is* $\mathbb{Q}$*-effective.*

Proof. As in the source paper, but using Theorem 23 for the finite generation. □

Definition 27. [CL, def 4.1] Let W a finite dimensional real vector space, $C \subset W$ a closed convex cone spanning W, and $v \in W$. The visible boundary of C from v is

$$V = \{w \in \partial C \mid [v, w] \cap C = \{w\}\}.$$

Corollary 28. [CL, 4.2] *(Kawamata's Rationality, Cone and Contraction Theorem) Let* (X, Δ) *be a klt pair of relative dimension* 2*, proper over a DVR* R *with perfect residue field* k *of characteristic* $p > 0$ *and perfect fraction field* K*. Assume either that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier and simple normal crossings over* R*, or is obtained after a finite number of steps of the minimal model program from such a* $\mathbb{Q}$*-divisor. Let* V *be the visible boundary of* $Nef(X/R)$ *from* $v_0 \in N^1(X)_{\mathbb{R}}$ *of the divisor* $K_X + \Delta$*. Then (1) every compact subset* $F \subset relint\ V$ *is contained in a union of finitely many supporting rational hyperplanes, and (2) every* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* X *with class in relint* V *is semiample.*

Proof. As in the cited theorem, noting it only requires the finite generation of Corollary 23, and Theorem 25. Part (2) is a consequence of Corollary 31. □

Lemma 29. [CL, Lemma 5.1]*Let* X *and* Y *be* $\mathbb{Q}$*-factorial projective surfaces over a perfect field* k*, let* $f : X \to Y$ *be a birational contraction, and let* $\tilde{f} : k(X) \approx k(Y)$ *be the induced isomorphism. Then:*

(1) $f_* div_X \varphi = div_Y \tilde{f}(\varphi)$

(2) for every geometric valuation Γ *on* $k(X)$ *and for every* $\varphi \in k(X)$ *we have* $mult_\Gamma(div_X \varphi) = mult_\Gamma\left(div_Y \tilde{f}(\varphi)\right)$

(3) if f *is an isomorphism in codimension one, then* $f_* : Div_{\mathbb{R}}(X) \to Div_{\mathbb{R}}(Y)$ *is an isomorphism, and for every* $D \in Div_{\mathbb{R}}(X)$*, the map* $\tilde{f}$ *restricts to the isomorphism* $H^0(X, D) \approx H^0(Y, f_* D)$*. (In this case,* X, Y *could be over a DVR* R *with perfect residue field).*

Proof. Follows easily from the cited Theorem. □

Lemma 30. [CL, 5.2] *Let* X,Y *be proper of relative dimension* 2*, over a DVR* R *with perfect residue field* k *of characteristic* $p > 0$ *and perfect fraction field* K *and* $f : X \to Y$ *a birational map which is an isomorphism in codimension one. Let* $\mathcal{C} \subset Div^{eff}_{\mathbb{R}}(X)$ *be a cone, and fix a geometric valuation* Γ *of* X*. Then the asymptotic order of vanishing* σ_Γ *is linear on* $\mathcal{C}$ *if and only if it is linear on* $f_*\mathcal{C} \subset Div^{eff}_{\mathbb{R}}(Y)$*.*

Proof. As in the cited theorem, substituting Lemma 29 as necessary. □

9. Big Abundance / R

Let Γ a geometric valuation on X. Define

$$\sigma_\Gamma = inf\{mult_\Gamma D' \mid D \sim_{\mathbb{R}} D' \geq 0\} \in \mathbb{R}.$$

As a result of Theorem 23, there is the following abundance theorem:

Corollary 31. [CL, 3.8] *Let* (X, Δ) *be a klt log smooth pair of relative dimension* 2 *over a DVR* R *with perfect residue field* k *of characteristic* $p > 0$ *and perfect fraction field* K*. Assume that* Δ *is big. If* $K_X + \Delta$ *is nef, then it is semi-ample.*

Proof. Let A an ample $\mathbb{Q}$-divisor on X such that $(X, \Delta + A)$ is klt. As Δ is big, it is possible to write $\Delta = \Delta' + A$ such that $(X, \Delta' + A)$ is klt. Thus the ring:

$$\mathcal{R} = R(K_X + \Delta, K_X + \Delta + A)$$

is finitely generated by Corollary 23, and *Supp* $\mathcal{R}$ contains a big divisor $(K_X + \Delta + A)$. Thus, any pseudo-effective

$$D \in \mathbb{R}_+ (K_X + \Delta) + \mathbb{R}_+ (K_X + \Delta + A)$$

is in *Supp* $\mathcal{R}$. By Theorem 25(2),

$$Supp\ \mathcal{R} = \mathbb{R}_+ (K_X + \Delta, K_X + \Delta + A).$$

For any $\epsilon > 0$, and any geometric valuation Γ on X,

$$\sigma_\Gamma (K_X + \Delta + \epsilon A) = 0$$

where, so that σ_Γ is zero on *Supp* $\mathcal{R}$, and thus the centre of Γ is not in $\mathbb{B} (K_X + \Delta)$ for any such Γ. □

Corollary 32. *Let X be a normal projective variety of dimension 2 (over a perfect field k) and let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Assume that the ring $R = R(X; D_1, ..., D_r)$ is finitely generated, and let Supp $R = \bigcup C_i$ be a finite rational polyhedral subdivision such that for every geometric valuation Γ of X, σ_Γ is linear on C_i, as in Theorem 25. Denote $\varphi : \sum \mathbb{R}_+ D_i \to N^1 (X)_\mathbb{R}$ the projection, and assume there exists k such that $C_k \cap \varphi^{-1} (Amp\ D) \neq \emptyset$. Then $C_k \subset Supp\ R \cap \varphi^{-1} (Nef\ \ X)$.*

Proof. As in the source, but applying Theorems 25 and 31. □

10. Contractions given by Extremal Rays / R

Lemma 33. [CL, 6.2]*Let (X, Δ) be a log smooth klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic*

$p > 0$ and perfect fraction field K. Suppose (X, Δ) is not nef, and A is a big $\mathbb{Q}$-divisor such that $(X, \Delta + A)$ is klt, and $K_X + \Delta + A$ is nef. Let λ be the nef threshold. Then $\lambda \in \mathbb{Q}^+$, and there is $R \subset \overline{NE}(X)$ with $(K_X + \Delta + \lambda A).R = 0$ and $(K_X + \Delta).R < 0$.

Proof. As in the cited theorem, but substituting Theorems 32 and 23 for their corresponding versions. □

11. Big Termination With Scaling

Theorem 34. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Suppose Δ is big and simple normal crossings over R. Let A be a sufficiently general ample $\mathbb{Q}$-divisor on X such that $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is nef. Then the $K_X + \Delta$-minimal model program with scaling of A terminates.*

Success is declared when there is a morphism $f : (X, \Delta) \to (X_n, \Delta_n)$ to a pair over a DVR such that on both geometric fibers $K_{X_n} + \Delta_n$ is nef. (Actually, applying [Liu, 5.3.24] and [Laz, 1.4.10], it suffices to show that the special fiber of the resulting model is nef).

Proof. This follows easily from [CL, 6.5], noting we have the required finite generation from Theorem 23. Let $\{\alpha_i\}$ be a sequence of numbers corresponding to a minimal model program with scaling defined on both fibers

$$(X_1, \Delta_1, \alpha_1 A_1) \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} (X_i, \Delta_i, \alpha_i A_i) \xrightarrow{f_i} \cdots$$

(α_i is the smallest positive real number such that $K_{X_i} + \Delta_i + \alpha_i A_i$ is nef on both fibers - although it suffices to check on the special fiber by Lemma 2). The claim is that the set of indices i is finite.

Choose r big divisors H_j arbitrarily close to $\Delta + \alpha$ for some norm on $N_1(X)$, such that *span* $\{K_X + H_j\}$ fills up all the dimensions of $N_1(X)$ and contains $K_X + \Delta + \alpha_1 A'$ on both fibers. Let C^1 be the $\mathbb{R}_+$-cone spanned by the $K_{X_1} + H_i$. Let Let C^i denote the proper transform of C^1 on X_i , H_j^i the proper transforms of H_j, and let

$$R^i = R\left(K_{X^i} + \Delta', K_{X^i} + H_1^i, ..., K_{X^i} + H_r^i\right)$$

$$\approx R^1 = R\left(K_X + \Delta, K_X + H_1, ..., K_X + H_r\right)$$

(The isomorphisms by Lemma 29(3)) which are all finitely generated by Theorem 23 under the identification to R^1.

By construction of the H^i, for each i, $G_i = K_{X_i} + \Delta_i' + \alpha_i A_i' \in int\ C^i$, and since G_i is nef, actually there are some ample divisors in $int\ C^i$. By Lemma 25, and since $Supp\ R^i = \bigcup C_\ell^i$ clearly contain big divisors, applying Theorem 26 (so that either G_i is $\mathbb{Q}$-Cartier so we can apply Theorem 25(3) or G_i is vacuously in $Supp\ R^i$) there are some ample divisors in C_ℓ^i for some ℓ. Thus, after applying Lemma 30 to ensure the linearity of geometric valuations, Theorem 32 applies to show that the C_ℓ^i consist of the pullbacks of nef divisors from X_i. Since there are only finitely many cones in the subdivision $Supp\ R_1 = \bigcup C_\ell^1$, eventually there will exist an index i_0, such that for $i > i_0$, the pullback of the nef cone of X_i lands in a C_ℓ^1 which is contained in the pullback of $Nef\left(X_{i'}\right)$ for some $i' \leq i_0$. After this point, applying the negativity lemma (c.f.

[Xu, 2.1] for positive characteristic and $dim \leq 3$) as in [CL, 6.5] gives a morphism. □

12. Reductions

Lemma 35. [HMX2014, 2.8.3] *Let (X, Δ) be a log smooth pair which is a scheme of dimension at most 3, with the coefficients of Δ belonging to $(0, 1]$, and with X projective. If (X, Δ) has a weak log canonical model, then there is a sequence $\pi : Y \to X$ of smooth blow ups of the strata of Δ such that if we write $K_Y + \Gamma = \pi^* (K_X + \Delta) + E$, where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_* \Gamma = \Delta$ and $\pi_* E = 0$ and if we write*

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma (Y, K_Y + \Gamma),$$

then $\mathbb{B}_- (Y, K_Y + \Gamma')$ contains no strata of Γ'. If Δ is a $\mathbb{Q}$-divisor, then Γ' is a $\mathbb{Q}$-divisor.

Proof. Follows easily from the cited theorem. □

Lemma 36. [GL, 2.3] *Let X/R a scheme which is projective of relative dimension 2 over either a DVR R with algebraically closed residue field and perfect fraction field or over a field. Let (X, Δ) be a klt pair and $\varphi : W \to X$ a log resolution of (X, Δ). Choose Δ_W so that $K_W + \Delta_W = \varphi^* (K_X + \Delta) + E$ with Δ_W and E effective $\mathbb{Q}$-Weil divisors with no common component. Let $F = \sum_{F_i \ \varphi-exceptional \ prime \ divisor} F_i$ and $\Delta_W^\epsilon = \Delta_W + \epsilon F$, then (W, Δ_W^ϵ) is an ϵ-log smooth model of (X, Δ). Then for $0 < \epsilon \ll 1$, a discrepancy negative (good) minimal model*

$\phi : (W, \Delta_W^{\epsilon}) \to \left(W_{min}, \Delta_{W_{min}}^{\epsilon}\right)$ with either ϕ a morphism or (X, Δ) log-smooth is also a (good) minimal model for (X, Δ).

Proof. Follows easily from [BCHM, 3.6.10, 3.6.11]. □

Lemma 37. [HMX2014, 5.3]*Let X/R a projective scheme of relative dimension 2 over a DVR R, with algebraically closed residue field and perfect fraction field. Let (X, Δ) be a log smooth dlt pair with X $\mathbb{Q}$-factorial and projective and Δ a $\mathbb{Q}$-divisor. If Φ is a $\mathbb{Q}$-divisor such that*

$$0 \leq \Delta - \Phi \leq N_{\sigma}\left(X, K_X + \Delta\right),$$

then steps of the $K_X+\Phi$-minimal model program are steps of the $K_X+\Delta$ minimal model program, and so termination for (X, Φ) implies termination for (X, Δ).

Proof. (Similar proof to the cited Theorem). Let $f : X \to Y$ the result of running a $(K_X + \Phi)$ minimal model program and let $Y \to Z$ be the ample model of $K_X + \Phi$. Let $\Delta_t = t\Delta + (1-t)\,\Phi$ so that if $0 < t \ll 0$, then f is also a $(K_X + \Delta_t)$-MMP. If C generates a $K_X + \Delta_t$-negative extremal ray, suppose that $(K_X + \Phi)\,.C > 0$. Applying Theorem 12 on the special fiber,

$$-4 \leq \left(K_X + \Delta\right).C < 0.$$

Then, following the inequalities in [HMX2014, 5.1], gives a contradiction if $0 < t < \frac{1}{5}$. Thus every step of the $K_X + \Delta_t$ with scaling of an ample divisor is $(K_X + \Phi)$-trivial, so after finitely many steps there is a model $g : X \to W$ contracting the components of

$N_\sigma(X, K_X+\Delta_t)$ (this holds with no changes from [HMX2014, 2.7.1]). Thus, as $supp\,(\Delta-\Phi)\subset supp\,N_\sigma(X,K_X+\Delta)=supp\,N_\sigma(X,K_X+\Delta_t)$, then $g_*(K_Y+\Phi)$ is semiample whenever $g_*(K_X+\Delta)$ is, and [HMX2014, 2.7.2] (which holds using only the log smooth resolution) implies that g is a minimal model of (X,Δ), which is good when (X,Φ) has a good minimal model.

□

13. Existence of Minimal Models Special Case

Lemma 38. [HMX2010, 3.1] *Let (X,Δ) be a $\mathbb{Q}$-factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p>0$ and perfect fraction field K. Assume that $\mathbb{B}_-(X_k, K_{X_k}+\Delta_k)$ contains no non-canonical centres of (X_k,Δ_k). Let $f: X\to Y$ be a step of the $(K_X+\Delta)$-MMP. If f is birational and V is a non-canonical centre of (X,Δ), then V is not contained in the indeterminacy locus of f, V_0 is not contained in the indeterminacy locus of f_0, and the induced maps $\phi: V\to W$ and $\phi_k: V_k\to W_k$ are birational, where $W=f(V)$. Let $\Gamma=f_*\Delta$. Then $\mathbb{B}_-(Y_0, K_{Y_0}+\Gamma_0)$ contains no non-canonical centres of (Y_0,Γ_0) (so we may repeat the process), and if V is a non-klt centre or $V=X$, then $\phi: V\to W$ and $\phi_0: V_0\to W_0$ are birational contractions. If f is a Mori fibre space, then f_0 is not birational.*

Proof. (Follows easily from the cited Theorem). Suppose f is birational. Let V be a non-canonical centre of (X,Δ). Let $g: X\to Z$ be the contraction of the extremal ray associated to f (so that $f=g$

unless f is a flip). Let $Q = g(V)$, and let $\psi : V \to Q$, be the induced morphism. As every component of V_k is a non-canonical centre of (X_k, Δ_k), then by hypothesis, components of V_k are not contained in $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$, thus ψ_k is defined at each such component, hence is birational. By upper-semicontinuity of the fibers of ψ, ψ is birational, and thus $\phi : V \to Q \to W$, and ϕ_k are both birational.

Now suppose V is a non-klt centre or $V = X$, the above holds in the first case, as non-klt centres are non-canonical. Comparing the discrepancies of the differents of adjunction for Δ_k and Γ_k as in the cited proof shows that ϕ_k, f_k, and thus ϕ are birational contractions. On the other hand, if f is a Mori fibre space, then, as the dimension of the fibers of $f : X \to Y$ are upper-semicontinuous, f_k is not birational. □

Theorem 39. [HMX2014, 3.2] *Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $\mathbb{B}_-(X_k, K_{X_k} + \Delta_k)$ contains no non-canonical centres of (X_k, Δ_k) and is log smooth Let $f : X \to Y$ be a step of the $(K_X + \Delta)$-MMP. Then the minimal model program with scaling terminates using only contractions.*

Proof. (Similar proof to the cited theorem). Let $f : X \to Y$ the $(K_X + \Delta)$-MMP with scaling of an ample divisor A. Let $\Gamma = f_*\Delta$ and $B = f_*A$. By construction, $K_Y + tB + \Gamma$ is nef for some $t > 0$. By Lemma 38, $f : X \to Y$ is a birational contraction and $f_k : X_k \to Y_k$ is a birational contractions from $(X_k, tA_k + \Delta_k)$. If $K_X + \Delta$ is not pseudo-effective, then for $t > 0$, the result of f is a Mori fiber space, and by Lemma 38, Y_k is covered by $K_{Y_k} + tB_k + \Gamma_k$-negative curves, contradicting bigness of $K_{X_k} + tA_k + \Delta_k$. Thus $K_X + \Delta$ is pseudo-effective, and given any $\epsilon > 0$, we may run the MMP until $t < \epsilon$. Now we conclude by Theorem 24. Letting ϵ be the constant given in that theorem, the minimal models for $(X_k, tA_k + \Delta_k)$ are all isomorphic $t \in [0, \epsilon]$. Thus, once $t < \epsilon$, any more steps in the minimal model program with scaling must be an isomorphism on the special fiber, and thus an isomorphism. Thus there exists a minimal model (Y, Γ) for (X, Δ). □

14. Existence of Minimal Models General Case

Theorem 40. *Let (X, Δ) be a klt pair of relative dimension 2, proper over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and log smooth over R. Then the minimal model of (X, Δ) exists.*

Proof. I start by repeating the reduction of [HMX2014, 6.1]. If necessary, extend R to be complete with algebraically closed residue field k, so the strata of Δ have irreducible fibers over R. Let $f_0 : Y_0 \to X_0$ be the birational morphism of Lemma 35. Under the log smooth hypothesis, and since the strata of Δ have irreducible fibers, and f_0 blows up strata of Δ_0, extend f_0 to a birational morphism $f : Y \to X/R$ which is a composition of smooth blow ups of strata of Δ. Write

$$K_Y + \Gamma = f^* (K_X + \Delta) + E$$

with $\Gamma \geq 0$ and $E \geq 0$, $f_*\Gamma = \Delta$, and $f_*E = 0$. Then (Y, Γ) is log smooth and the fibres of components of Γ are irreducible. By Lemma 36, (X, Δ) has a minimal model if (Y, Γ_ϵ) has a good minimal model, where $(Y, \Gamma_\epsilon) = (Y, \Gamma + \epsilon F)$ is the ϵ-log smooth model of (X, Δ) with F the sum of f-exceptional divisors and $0 < \epsilon \ll 1$.

Replace (X, Δ) by (Y, Γ_ϵ) and set $\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma (X_k, K_{X_k} + \Delta_k)$ so that $\mathbb{B}_- (X_k, K_{X_k} + \Theta_k)$ contains no strata of Θ_k. Let $0 \leq \Theta \leq \Delta$ be the unique divisor such that $\Theta_k = \Theta|_{X_k}$. Let H relatively ample such that $(X, \Theta + H)$ is log smooth over R, and $K_X + \Theta + H$ is big. There is a commutative diagram:

$$\begin{array}{ccc} \pi_*\mathcal{O}_X (m (K_X + \Theta) + H) & \longrightarrow & \pi_*\mathcal{O}_X (m (K_X + \Delta) + H) \\ \downarrow & & \downarrow \\ H^0 (X_k, \mathcal{O}_{X_k} (m (K_{X_k} + \Theta_k) + H_k)) & \longrightarrow & H^0 (X_k, \mathcal{O}_{X_k} (m (K_{X_k} + \Delta_k) + H_k)) \end{array}$$

with surjective columns by Theorem 17 and [Liu, 5.3.20(b)], and with the bottom row an isomorphism. Applying Nakayama's lemma gives an isomorphism on the top row, so that $\Theta \geq \Delta - \Delta \wedge N_\sigma (X, K_X + \Delta)$. Again applying Theorem 17, gives $\Theta = \Delta - \Delta \wedge N_\sigma (X, K_X + \Delta)$. Thus $\Delta - \Theta \leq N_\sigma (X, K_X + \Delta)$, so by Lemma 37, it suffices to find a minimal model for (X, Θ). Replacing (X, Δ) by (X, Θ), it suffices to assume that $\mathbb{B}_- (X_k, K_{X_k} + \Delta_k)$ contains no strata of Δ_0. Letting A be an ample divisor, we run the minimal model program with scaling of A. Now the assumptions of Theorem 39 apply, so we know that (X, Θ) has a minimal model. □

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