Adjoint Rings on Arithmetic Threefolds

August 13, 2015

Abstract

Finite generation of various types of adjoint rings of a Arithmetic threefold over a DVR and consequences. Updates posted at divisibility.wordpress.com

Andrew Egbert

1 Intro

In [Liu], an arithmetic surface is defined as a regular fibered surface over a Dedekind scheme of dimension 1, in terms of dimensions, the generic and special fibers are actually curves. Thus in this article, an arithmetic threefold will mean an integral, projective, flat S scheme $\pi : X \to S$ of dimension 3, with generic fiber X_η and closed field X_s surfaces where S is *Spec* R for a Dedekind scheme S. This article will be concerned with the case that R is a Discrete valuation ring, i.e. a local Dedekind domain.

In order of increasing difficulty, the specific subject matter is the properties N_p (which concern adjoint rings associated to divisors $K_X + A$ where A is ample); the finite generation of adjoint rings $R(K_X + \Delta, K_X + \Delta + A)$ where A is ample and (X, Δ) is a log smooth KLT pair, and finally, the finite generation of the adjoint rings $R(K_X + \Delta)$ where (X, Δ) is a KLT log smooth pair, as a consequence of the existence of minimal models for such pairs.

Recall the property N_p:

Definition 1. ([EL], M_F and Property N_p) Let X be a projective variety, F a globally generated vector bundle on X. Let M_F be the kernel

$$0 \to M_F \to H^0(F) \otimes \mathcal{O}_X \to F \to 0 \quad \star$$

For L ample and globally generated line bundle, L satisfies property N_p if

$$H^1\left(L^{\otimes r}\right) = 0$$

for all $r \geq 1$ and

$$H^1\left(M_L^{p+1} \otimes L^s\right) = 0$$

for all $0 \leq p' \leq p$.

Tensoring the exact sequence $\star$ with L, we have

$$\cdots \to H^0(L) \otimes H^0(L) \to H^0(2L) \to H^1(M_L \otimes L) \to H^1(L) \otimes H^0(L) \to \cdots$$

so in order to show property N_0 for $L = K_X + 4B$, the second condition can be replaced by showing that the multiplication map

$$H^0(L)^2 \to H^0(2L) \quad \star\star$$

is surjective. Note that this surjectivity is equivalent to the adjoint ring $R(L)$ being generated in degree 1.

In characteristic 0, Gallego, Purnaprajna, Hanumanthu, and Banagere **[BH, GP, GP1, GP2]** have proved various results along the lines of the following:

Theorem 2. *[P, 3.14] Let S be a surface of general type, let A be an ample line bundle and let $m = \left[\frac{(A.(K_S+4A)+1)^2}{2A^2}\right]$. Let $L = K_S \otimes A^{\otimes n}$. If $n \geq 2m$, then L satisfies property N_0 and even N_1.*

The first aim of this article is to prove a theorem similar to the above, but on a surface over a DVR R in positive or mixed characteristic. This article is concerned with the basic techniques, and presumably various tricks from the characteristic 0 case employed by the aforementioned authors could be used to obtain sharper results. Specifically, one would like to prove the following:

Theorem 3. *Let X/R be a smooth surface of general type, let A be an ample line bundle, and let $L = K_X + nA$. There is a computable constant M such that if $n > M$, then L satisfies N_0 and N_1.*

Thus:

Corollary 4. *Let X/R be a smooth surface with K_X ample, then the canonical ring is generated in degree M where M is a computable constant.*

Another consequence of these techniques is the following (explain more)

Theorem 5. *Let (X,Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k and perfect fraction field K. Assume that $K_X+\Delta$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over R. Then the adjoint ring $R(K_X+\Delta, K_X+\Delta+A)$ is finitely generated where A is an ample divisor.*

Proof. (Sketch) Let m,n be integers such that $m>n+2\gg 0$. C.f. [CZ, 2.11], one must show surjectivity of the following multiplication map

$$H^0(ma(K_X+\Delta+A)+nb(K_X+\Delta)-a(K_X+\Delta+A)=D)$$
$$\otimes H^0(a(K_X+\Delta+A)=D')\to H^0(D+D').$$

Choose a an integer large enough so that [Eg, cor 28.] applies to the second module in the above product. Note that

$$ma(K_X+\Delta+A)+nb(K_X+\Delta)-a(K_X+\Delta+A)\sim$$

$$((m-1)a+nb)(K_X+\Delta)+(m-1)aA\sim$$
$$((m-1)a+nb)(K_X+\Delta)+((m-1)a+nb)dA'$$

$d=\frac{(m-1)a}{((m-1)a+nb)}$ so choose b large enough that [Eg, cor 28.] applies to the first module in the above product. By the Nakayama's lemma surjectivity of the above map can be checked on the special fiber (possibly after base change so the residue field is algebraically closed).

To check surjectivity on the special fiber, we check finite generation of $R(K_{X_k}+\Delta_k, K_{X_k}+\Delta_k+A_k)$ (and then possibly increase m,n). [Tan] applies to give termination of the minimal model program (hence the minimal model program with scaling of an ample divisor A terminates), thus c.f. [HX2013, lemma 2.7] there exists a rational number $t_0>0$ and a birational contraction $\phi: X_k\to X'_k$ which is a $K_{X_k}+\Delta_k+tA_k$ good minimal model for all $0\le t\le t_0$, and so, as nef implies semiample by [Tan], then c.f.[H2009, 5.5] gives that semi-ample divisors in arbitrary characteristic have finitely generated Cox ring, the ring $R(X_k, K_{X_k}+\Delta_k, K_{X_k}+\Delta_k+A_k)$ is finitely generated. □

As a corollary to the above, c.f. [CL],

Corollary 6. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k and perfect fraction field K. Assume that $K_X + \Delta$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over R. Then the minimal model of (X, Δ) exists over R. Also, nefness of $K_X + \Delta$ implies semi-ampleness on both fibers.*

Proof. 4 (sketch) (Essentially [CL, 6.8]) Suppose A is a sufficiently general ample $\mathbb{Q}$-divisor such that $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is nef. Let $\{\lambda_i\}$ be a sequence of positive real numbers corresponding to a minimal model program with scaling of $A = A_1$. Note that

$$R_i = R\left(X_i; K_{X_i} + \Delta_i, K_{X_i} + \Delta_i + \lambda_i A_i\right)$$

$$\approx R\left(X_1; K_{X_1} + \Delta_1, K_{X_1} + \Delta_1 + \lambda_i A_1\right)$$

are all finitely generated by Theorem 5. The remainder of the proof is given in section 4. □

2 Background

The following results are necessary.

Theorem 7. *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X. Assume that either*

(1) $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or

(2) X is of general type with

$p \geq 3$ and $(D^2) > vol(X)$ or

$p = 2$ and $(D^2) > max\{vol(X), vol(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then

$$H^i\left(X, \mathcal{O}_X\left(K_X + D\right)\right) = 0$$

for all $i > 0$.

Theorem 8. *[DF, 4.11] (Effective Base-point Freeness) Let X be a surface of general type and let D be a big and nef divisor on X. Then the following holds.*

If $p \geq 3$ *and* $D^2 > vol(X) + 4$ *and* $|K_X + D|$ *has a base point at* $x \in X$*, then there exists a curve* C *such that* $D.C \leq 1$*.*

If $p = 2$ *and if* $D^2 > vol(X) + 6$ *and* $|K_X + D|$ *has a base point at* $x \in X$*, then there exists a curve* C *such that* $D.C \leq 7$*.*

Theorem 9. *[DF, 1.2] Let* D *and* B *be respectively an ample divisor and a nef divisor on a smooth surface* X *over an algebraically closed field* k*, with char* $k = p > 0$*. Then* $mD - B$ *is very ample for any*

$$m > \frac{2D.(H+B)}{D^2}\left(\left(K_X + 2D\right).D + 1\right)$$

where

$H = K_X + 4D$ *if* X *is neither quasi-elliptic with* $\kappa(X) = 1$ *nor of general type*

$H = K_X + 8D$ *if* X *is quasi-elliptic with* $\kappa(X) = 1$ *and and* $p = 3$*;*

$H = K_X + 19D$ *if* X *is quasi-elliptic with* $\kappa(X) = 1$ *and* $p = 2$*;*

$H = 2K_X + 4D$ *if* X *is of general type and* $p \geq 3$*;*

$H = 2K_X + 19D$ *if* X *is of general type and* $p = 2$*.*

Theorem 10. *(Nakayama's Lemma) Let* $M_0 = M/\mathfrak{m}M$ *, where* M *is a module over a dvr* R*. If* $\varphi : M \to N$ *is a homomorphism of* R*-modules such that* $\varphi_0 : M_0 \to N_0$ *is surjective, then* φ *is surjective.*

The following theorem of Mumford is important to show surjectivity of multiplication maps.

Theorem 11. *([M], Theorem 2) Suppose* L *is an ample invertible sheaf on a variety* X *such that* $\Gamma(L)$ *has no base points. Suppose* $\mathcal{F}$ *is a coherent sheaf on* X *such that*

$$H^i\left(\mathcal{F} \otimes L^{-i}\right) = 0,$$

$i \geq 1$*. Then*

$$H^i\left(\mathcal{F} \otimes L^j\right) = 0,\ i + j \geq 0,\ i \geq 1$$

and

$$S\left(\mathcal{F} \otimes L^i, L\right) = 0,\ i \geq 0$$

where $S(A, B)$ *stands for the cokernel of the map given by multiplying global sections.*

the induction lemma

Lemma 12. *([GP], 1.4.1) Let E and $L_1, ..., L_r$ be coherent sheaves on a variety X. Consider the map*

$$H^0(E) \otimes H^0(L_1 \otimes \cdots \otimes L_r) \xrightarrow{\psi} H^0(E \otimes L_1 \otimes \cdots \otimes L_r)$$

and the maps

$$H^0(E) \otimes H^0(L_1) \xrightarrow{\alpha_1} H^0(E \otimes L_1),$$

$$H^0(E \otimes L_1) \otimes H^0(L_2) \xrightarrow{\alpha_2} H^0(E \otimes L_1 \otimes L_2)$$

$$\cdots$$

$$H^0(E \otimes L_1 \otimes \cdots \otimes L_{r-1}) \otimes H^0(L_r) \xrightarrow{\alpha_r} H^0(E \otimes L_1 \otimes \cdots \otimes L_r).$$

If $\alpha_1, ..., \alpha_r$ are surjective, then ψ is also surjective.

Theorem 13. *[CP2014, 1.1] Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*
(i) X' is everywhere regular
(ii) π induces an isomorphism $\pi^{-1}(Reg\ X) \approx Reg\ X$
(iii) $\pi^{-1}(Sing\ X)$ is a strict normal crossings divisor on X'.

Definition 14. (Asymptotic Order of Vanishing) Let X be a normal projective variety, D an effective $\mathbb{R}$ divisor and Γ a geometric valuation of X. Define $\sigma_\Gamma(D) = inf\{mult_\Gamma D' | D \sim_{\mathbb{R}} D' \geq 0\} \in \mathbb{R}$.

3 Proof Theorem 3

First I explore the properties N_0 and N_1. R is a discrete valuation ring with residue field k and fraction field K.

Lemma 15. *Let X/R be a smooth surface of general type, A an ample $\mathbb{Z}$-divisor, and R has perfect residue field k and perfect fraction field K. Suppose that $L = K_X + A$ such that $K_X + A$ is nef and big. Then*

$$H^1(L^k) = 0$$

for all k.

Proof. Note that $(K_X + A)^2 > K_X^2 = vol\,(X)$, so Terakawa's vanishing theorem applies to

$$K_X + \underbrace{(A + (k-1)\,(K_X + A))}_{D}$$

on X_k, and by semicontinuity/ Nakayama's Lemma, the result holds on the generic fiber as well, since a basis for cohomology module on the special fiber lifts to a set of generators on the local ring.

□

Lemma 16. *Let X/R be a smooth surface of general type, A an ample $\mathbb{Z}$-divisor, and R has perfect residue field k and perfect fraction field K. Suppose that $L = K_X + A$ so that $K_X + A$ is nef and big. Then surjectivity of the map*

$$H^0\left(L|_{X_k}\right) \otimes H^0\left(L^{\otimes m}|_{X_k}\right) \to H^0\left(L|_{X_k}^{\otimes 2}\right)$$

for all m implies property N_0 for L on X/R.

Proof. As $X_k \sim 0$ on X/R, then there is an exact sequence in cohomology

$$0 \to \mathcal{O}_X\,(m\,(K_X + A)) \to \mathcal{O}_X\,(m\,(K_X + X_k + A))$$
$$\to \mathcal{O}_{X_k}\,(m\,(K_{X_k} + A_k)) \to 0.$$

Taking cohomology and appliying lemma 15 gives a surjection:

$$H^0\,(m\,(K_X + A)\,/R) \to H^0\,(m\,(K_{X_k} + A_k))$$

for any m. Now applying Nakayama's lemma to the diagram

$$\begin{array}{ccc} H^0\,(m\,(K_X + A)\,/R) \otimes H^0\,(K_X + A/R) & \longrightarrow & H^0\,((m+1)\,(K_X + A)\,/R) \\ \downarrow & & \downarrow \\ H^0\,(m\,(K_{X_k} + A_k)) \otimes H^0\,(K_{X_k} + A_k) & \longrightarrow & H^0\,((m+1)\,(K_{X_k} + A_k)) \end{array}$$

gives that surjectivity of the bottom map implies surjectivity of the top map. From the exact sequence:

$$\to H^0\left(L^{\otimes m}\right) \otimes H^0\,(L) \to H^0\left(L^{\otimes m+1}\right)$$
$$\to H^1\left(M_L^1 \otimes L^m\right) \to H^1\left(L^{\otimes m}\right) \otimes H^0\,(L) \to$$

then applying lemma 15 it suffices to show surjectivity of the top map in the above diagram for each $m \geq 1$. □

Lemma 17. *Let X/R be a smooth surface of general type, A an ample $\mathbb{Z}$-divisor, and R has perfect residue field k and perfect fraction field K. Suppose that $r > 3$ and define $L_r = K_{X_k} + rA_k$ so that $K_{X_k} + A_k$ and $A_k = lA'_k$ are base-point free and $A_k - K_X$ is nef and big with $(A_k - K_{X_k})^2 > K^2_{X_k}$ (such an r and l are computable based on the various intersection numbers of K_{X_k} and A_k over an algebraically closed field of characteristic $p > 0$ using theorems 8, 9). Then*

$$H^0\left(L_r|_{X_k}\right) \otimes H^0\left(L_r^{\otimes m}|_{X_k}\right) \to H^0\left(L_r^{\otimes m+1}|_{X_k}\right)$$

is surjective for all m.

Proof. Suppose first that $m = 1$, the other cases will follow similarly. By Lemma 12, it suffices to show that

$$H^0\left(K_{X_k} + rA_k\right) \otimes H^0\left(K_{X_k} + A_k\right) \to H^0\left(2K_{X_k} + (r+1)A_k\right),$$

$$H^0\left(2K_{X_k} + (r+1)A_k\right) \otimes H^0\left(A_k\right) \to H^0\left(2K_{X_k} + (r+2)A_k\right),$$

$$\cdots,$$

$$H^0\left(2K_{X_k} + (2r-1)A_k\right) \otimes H^0\left(A_k\right) \to H^0\left(2K_{X_k} + 2rA_k\right)$$

are surjective. By Lemma 11, the first surjectivity follows from the vanishing of

$$H^1\left((r-1)A_k\right) = H^1\left(K_{X_k} + (r-1)A_k - K_{X_k}\right)$$

and

$$H^2\left((r-2)A_k - K_{X_k}\right) = H^0\left(2K_{X_k} - (r-2)A_k\right).$$

The first vanishing holds by Terakawa's Theorem 7 since by assumption $A_k - K_{X_k}$ is nef and big with $(A_k - K_{X_k})^2 > K^2_{X_k}$. The second vanishing theorem holds since $2(A_k - K_{X_k}) \geq 0$ and $r > 3$. The other surjectivities follow similarly.

If $m = 2$, then similar logic applies to the maps

$$H^0\left(2\left(K_{X_k} + rA_k\right)\right) \otimes H^0\left(K_{X_k} + A_k\right) \to H^0\left(3K_{X_k} + (2r+1)A_k\right),$$

$$H^0\left(3K_{X_k} + (2r+1)A_k\right) \otimes H^0\left(A_k\right) \to H^0\left(3K_{X_k} + (2r+2)A_k\right),$$

$$\cdots,$$

$$H^0\left(3K_{X_k} + (3r-1)A_k\right) \otimes H^0\left(A_k\right) \to H^0\left(3K_{X_k} + 3rA_k\right)$$

and the necessary vanishings are slightly easier. □

The above proposition completes the N_0 portion of Theorem 3.

Claim 18. N_1 also holds for L_r (possibly slightly enlarging l and r as defined in the previous lemma).

Proof. We must show, in addition to what was proved in the first claim, that $H^1\left(M_L^{\otimes 2}\otimes L^{\otimes b}\right)=0$ for all $b\geq 1$. I'll start with the $b=1$ case for simplicity. There is an exact sequence:

$$H^0\left(M_L^{\otimes 2}\otimes L\right)\to H^0\left(M_L\otimes L\right)\otimes H^0\left(L\right)$$
$$\to H^0\left(M_L\otimes 2L\right)\to H^1\left(M_L^2\otimes L\right)\to H^1\left(M_L\otimes L\right)\otimes H^0\left(L\right)\to ..$$

it suffices to show surjectivity of the second map. Using Lemma 12 and Lemma 11 as in the argument for N_0, it suffices to show (1) that

$$H^1\left(M_L\otimes L\otimes\mathcal{O}_X\left(-K_X-A\right)\right)=0,$$
$$H^1\left(M_L\otimes L\otimes\mathcal{O}_X\left(K_X+A-A\right)\right)=0,$$
$$\cdots,$$
$$H^1\left(M_L\otimes L\otimes\mathcal{O}_X\left(K_X+(r-1)A-A\right)\right)=0,$$

and these follow as in the N_0 case after possibly enlarging r,l; and (2) also that

$$H^2\left(M_L\otimes L\otimes\mathcal{O}_X\left(-K_X-2A\right)\right)=0,$$
$$H^2\left(M_L\otimes L\otimes\mathcal{O}_X\left(K_X+A-2A\right)\right)=0,$$
$$\cdots,$$
$$H^2\left(M_L\otimes L\otimes\mathcal{O}_X\left(K_X+(r-1)A-2A\right)\right)=0.$$

Continuing the above exact sequence to second cohomology gives:

$$\to H^1\left(M_L\otimes L\right)\otimes H^0\left(L\right)\to H^1\left(M_L\otimes L^{\otimes 2}\right)$$
$$\to H^2\left(M_L^{\otimes 2}\otimes L\right)\to H^2\left(M_L\otimes L\right)\otimes H^0\left(L\right)\to$$

so it suffices to show both $H^1\left(M_L\otimes L^{\otimes 2}\right)=0$ and $H^2\left(M_L\otimes L\right)=0$. Vanishing of $H^1\left(M_L\otimes L^{\otimes 2}\right)$ follows from property N_0 and for the second vanishing, consider

$$\to H^1\left(L\right)\otimes H^0\left(L\right)\to H^1\left(L^{\otimes 2}\right)\to H^2\left(M_L\otimes L\right)\to H^2\left(L\right)\otimes H^0\left(L\right)\to .$$

Since $H^1\left(L^{\otimes 2}\right)=0$ by Lemma 15, and then it suffices to show vanishing of $H^2\left(K_{X_k}+rA_k\right)\approx H^0\left(K_{X_k}-\left(K_{X_k}-rA_k\right)\right)$. This is clear. For $b>1$, the above arguments are similar (easier since $L^{\otimes 2}$ is more positive than just L). ∎

4 Proof of Corollary 6

The ideas in this section, except for some small modifications, are from the paper "New Outlook on Minimal Model Program II" by Corti and Lazic. The point of repeating them here is that the Conjecture 1.2 mentioned in that paper seems? to hold in the case of an arithmetic threefold germ (c.f. Theorem 4), but there are some slight, but important, modifications necessary in order for the rest of these ideas to (hopefully) hold in this scenario.

The next three results are restated for a perfect field k and dimension 2.

Theorem 19. *([CL, 3.5] Let X be a normal projective variety of dimension 2 (over a perfect field k), and let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier divisors on X. Assume that $R(X; D_1, ..., D_r)$ is finitely generated, and let $D : \mathbb{R}^r \ni (\lambda_1, ..., \lambda_r) \mapsto \sum \lambda_i D_i \in Div_{\mathbb{R}}(X)$ be the tautological map.*

(1) The support of R is a rational polyhedral cone.

(2) If Supp R contains a big divisor and $D \in \sum \mathbb{R}_+ D_i$ is pseudoeffective, then $D \in$ Supp R.

(3) There is a finite rational polyhedral subdivision Supp $R = \bigcup C_i$ such that σ_Γ is a linear function on C_i for every geometric valuation Γ of X. Furthermore, there is a coarsest subdivision with this property, in the sense that, if i and j are distinct, there is at least one geometric valuation Γ of X such that (the linear extensions of) $\sigma_\Gamma|_{C_i}$ and $\sigma_\Gamma|_{C_j}$ are different.

(4) There is a finite index subgroup $\mathbb{L} \subset \mathbb{Z}^r$ such that for all $n \in \mathbb{N}^r \cap \mathbb{L}$, if $D(n) \in$ Supp R, then $\sigma_\Gamma(D(n)) = mult_\Gamma |D(n)|$.

Proof. 3.5(1, 2) hold the same as in the source paper, noting that [ELM, 4.7] is stated for an arbitrary Noetherian scheme, and changing $\mathbb{C}$-algebra to k-algebra in the necessary parts of 4.1 in that paper. □

Theorem 20. *([CL, 3.6]) Let (X, Δ) be a projective klt pair of dimension 2 (over a perfect field k) where Δ is big. If $K_X + \Delta$ is pseudoeffective, then it is $\mathbb{Q}$-effective.*

Proof. As in the source paper, but using Theorem 5 for the finite generation. □

Theorem 21. *Let X be a normal projective variety of dimension 2 (over a perfect field k) and let $D_1, ..., D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Assume that the ring $R = R(X; D_1, ..., D_r)$ is finitely generated, and let $Supp\ R = \bigcup C_i$ be a finite rational polyhedral subdivision such that for every geometric valuation Γ of X, σ_Γ is linear on C_i, as in Theorem 19. Denote $\varphi : \sum \mathbb{R}_+ D_i \to N^1(X)_\mathbb{R}$ the projection, and assume there exists k such that $C_k \cap \varphi^{-1}(Amp\ D) \neq \emptyset$. Then $C_k \subset Supp\ R \cap \varphi^{-1}(Nef\ X)$.*

Proof. As in the source, but applying Theorem 19 and using abundance for this situation c.f. [Tan]. □

Theorem 22. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair of relative dimension 2, log smooth over a DVR R with algebraically closed residue field k. Let A be sufficiently general ample $\mathbb{Q}$-divisor on X such that $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is nef. The minimal model program with scaling of A terminates.*

The proof will be given in two steps. The first reduction is to the case with big boundary. Success is declared when there is a morphism $f : (X, \Delta) \to (X_n, \Delta_n)$ to a pair over a DVR such that on both geometric fibers $K_{X_n} + \Delta_n$ is nef.

Proof. (Essentially just a slightly modified version of [CL, 6.5,6.8]) The first goal is to reduce to the case where Δ is big. Thus suppose, for now, that $K_X+\Delta$ is pseudo-effective, otherwise a $K_X+\Delta$-contraction is a $K_X+\Delta+\epsilon A$-contraction for A ample and $0 < \epsilon \ll 1$, so we may assume $\Delta' = \Delta + \epsilon A$ which is a big boundary.

Consider a minimal model program with scaling of $A_1 = A$ run (slightly different from usual) as follows. Let

$$\lambda = inf\left\{t \geq 0 \mid K_X + \Delta + tA_1\ is\ nef\ on\ both\ fibers\right\}.$$

If $\lambda = 0$, then we are done. Otherwise, there exists a $(K_X + \Delta)$-negative extremal ray C on one or both fibers such that $(K_X + \Delta + \lambda A).C \geq 0$ on both fibers and is 0 on at least one of them. Take the contraction on $(X, \Delta)/R$ given by the main Theorem of [Eg] and replace $(X, \Delta)/R = (X_1, \Delta_1)$ by the corresponding flip or contraction $\phi : (X_1, \Delta_1) \to (X_2, \Delta_2)/R$. Let $A_2 = \phi_* A_1$ and $\Delta_2 = \phi_* \Delta_1$. Then $K_{X_2} + \Delta_2 + \lambda A_2$ is nef on both fibers, so noting that:

$$R_i = R\left(X_i, K_{X_i} + \Delta_i, K_{X_i} + \Delta_i + \lambda_i A_i\right)$$

$$\approx R\left(X_1, K_{X_1}+\Delta_1, K_{X_1}+\Delta_1+\lambda_i A_1\right)$$

which are finitely generated by Theorem 5, we repeat the process, since we can argue the contraction exists on X_1, which is log-smooth. Theorem 19 implies that

$$Supp\ R_{i_0,k\ or\ K}$$
$$=\mathbb{R}_+\left(K_{X_{i_0}}+\Delta_{i_0}\right)+\mathbb{R}_+\left(K_{X_{i_0}}+\Delta_{i_0}+\lambda_{i_0}A_{i_0}\right)$$
$$=\bigcup \mathcal{C}_j^{i_0}$$

a rational polyhedral subdivision on both fibers. (The first equality since $K_X+\Delta+\lambda A$ is big so any pseudo-effective divisor in the cone of

$$\left\{K_{X_{i_0}}+\Delta_{i_0}, K_{X_{i_0}}+\Delta_{i_0}+\lambda_{i_0}A_{i_0}\right\}$$

is in $Supp\ R$). Let $C^i_{j,k}$,$C^i_{j,K}$ denote the proper transform of $C^{i_0}_{j,k}$ and $C^{i_0}_{j,K}$ respectively if $i\geq i_0$. By Lemma 19, for each geometric valuation $\sigma_{\Gamma,k}$, $\sigma_{\Gamma,K}$ is linear on $C^{i_0}_{j,k}$ and $C^{i_0}_{j,K}$ respectively.

Note it is impossible that $D_i=K_{X_i}+\Delta_i+\lambda_i A_i\in int\ \mathcal{C}^i_j$ on both fibers at once: By non-vanishing, D_i is semiample over k, K, and $\sigma_{\Gamma,k},\sigma_{\Gamma,K}$ are linear and non-negative on $C^i_{j,k}$,$C^i_{j,K}$ by Lemma 19, thus identically zero, thus by finite generation on each fiber (c.f. [Tan]) any divisor in $C^i_{j,k}, C^i_{j,K}$ are nef, so λ_i is not the smallest such that $K_{X_i}+\Delta_i+\lambda_i A_i$ is nef on both fibers simultaneously, contradicting the construction. As there are only finitely many boundary rays in both $C^{i_0}_{j,k}$ and $C^{i_0}_{j,K}$ together, and $K_{X_i}+\Delta_i+\lambda_i A_i$ must be at a boundary ray on one of the fibers, the set $\{\lambda_i\}$ is finite.

Now the argument follows [CL, 6.5], working on both fibers at once. Letting λ be the minimum of the λ_i, if $\lambda\neq 0$, then it suffices to consider the (modified as above to both fibers) minimal model program with scaling of $(\lambda-\epsilon)A_1=A'$ for $K_X+\Delta'$, where $\Delta'=\Delta+\epsilon A$ for $0<\epsilon<\lambda$, so we have reduced to the case with big boundary. Set $\alpha_1=1$ so that

$$K_X+\Delta'+\alpha_1 A_1$$

$$=K_X+\Delta+\epsilon A+(\lambda-\epsilon)A=K_X+\Delta+\lambda A$$

is nef on both fibers, and let $\{\alpha_i\}$ be a sequence of numbers corresponding to a minimal model program defined as before (α_i is the smallest positive real number such that $K_{X_i}+\Delta_i+\alpha_i A_i$ is nef on both fibers). The claim is that the set of indices i is finite.

Choose r big divisors H_j arbitrarily close to $\Delta + \alpha$ for some norms on $N_1(X_k), N_1(X_K)$, and such that the *span* $\{K_{X_1} + H_j\}$ fills up all the dimensions of $N_1(X_k)$ and $N_1(X_K)$ simultaneously and contains $K_X + \Delta + \alpha_1 A'$ on both fibers. Let C_k^1, C_K^1 be the $\mathbb{R}_+$-cone spanned by the $K_{X_1} + H_i$ on X_k, X_K respectively. Let C_k^i, C_K^i denote the proper transforms, $H_j^{i,k}$ or $H_j^{i,K}$ the proper transforms of $H_j|_{X_k}$, $H_j|_{X_K}$ and let

$$R_k^1 \approx R_k^i = R\left(K_{X_k^i} + \Delta_k', K_{X_k^i} + H_1^{i,k}, ..., K_{X_k^i} + H_r^{i,k}\right),$$

$$R_K^1 \approx R_K^i = R\left(K_{X_K^i} + \Delta_K', K_{X_K^i} + H_1^{i,K}, ..., K_{X_K^i} + H_r^{i,K}\right).$$

which are all finitely generated under the identification to R_k^1, R_K^1 as in Theorem 5.

By construction of the H^i, for each i, $G_i = K_{X_i} + \Delta_i' + \alpha_i A_i' \in int\ C^i$ on both fibers, and since G_i is nef on both fibers, actually there are some ample divisors in $int\ C^i$. By Lemma 19, and since $Supp\ R_k^i = \bigcup C_{\ell_k}^i$, $Supp\ R_K^i = \bigcup C_{\ell_K}^i$ clearly contain big divisors, applying Theorem 20 (so that either G_i is $\mathbb{Q}$-Cartier so we can apply 3.5(3) or G_i is vacuously in $Supp\ R_{K,k}^i$) there are some ample divisors in $C_{\ell_k}^i$, $C_{\ell_K}^i$ for some ℓ_k,ℓ_K. Thus using Theorem 21, $C_{\ell_k}^i$, $C_{\ell_K}^i$ consist of the pullbacks of nef divisors from X_i. Focusing on X_k, since there are only finitely many cones in the subdivision $Supp\ R_1 = \bigcup C_{\ell_k}^1$, eventually there will exist an index i_0, such that for $i > i_0$, the pullback of the nef cone of X_i lands in a $C_{\ell_k}^1$ which is contained in the pullback of $Nef\,(X_{i'})$ for some $i' \leq i_0$. After this point, applying the negativity lemma (c.f. [Xu, 2.1] for positive characteristic and $dim \leq 3$) as in [CL, 6.5] gives a morphism on the special fiber. By construction of the above minimal model program with scaling, then for $i > i_0$, and applying the same logic to X_K^i, gives that eventually these are all also morphisms. □

References

[BH] Banagere, P., and Krishna Hanumanthu. "Syzygies of surfaces of general type." Geometriae Dedicata 167.1 (2013): 123-149.

[CL] Corti, Alessio, and Vladimir Lazić. "New outlook on the minimal model program, II." Mathematische Annalen 356.2 (2013): 617-633.

[CP2009] Cossart, Vincent, and Olivier Piltant. "Resolution of singularities of threefolds in positive characteristic II." Journal of Algebra 321.7 (2009): 1836-1976.

[CP2014] Cossart, Vincent, and Olivier Piltant. "Resolution of Singularities of Arithmetical Threefolds II." arXiv preprint arXiv:1412.0868 (2014).

[CZ] Cascini, Paolo, and DEQI ZHANG. "Effective finite generation for adjoint rings." arXiv preprint arXiv:1203.5204 (2012).

[DF] Di Cerbo, Gabriele, and Andrea Fanelli. "Effective Matsusaka's Theorem for surfaces in characteristic p." arXiv preprint arXiv:1501.07299 (2015).

[Eg] Egbert, Andrew. "Invariance of Plurigenera and Finite Generation for Log Surfaces of General Type in Mixed Characteristic." available on divisibility.wordpress.com (see surfdiop1.5, 2.9 for e.g.)

[EL] Ein, Lawrence, and Robert Lazarsfeld. "Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension." Inventiones mathematicae 111.1 (1993): 51-67.

[ELM] Ein, Lawrence, et al. "Asymptotic invariants of base loci." Annales de l'institut Fourier. Vol. 56. No. 6. 2006.

[G] Gongyo, et al. https://www.dpmms.cam.ac.uk/~cb496/conf-chula/talk-okawa.pdf

[GP] Gallego, Francisco Javier, and B. P. Purnaprajna. "Projective normality and syzygies of algebraic surfaces." Journal für die reine und angewandte Mathematik (Crelles Journal) 1999.506 (1999): 145-180.

[GP1] Gallego, F. J., and B. P. Purnaprajna. "Syzygies of projective surfaces: an overview." JOURNAL-RAMANUJAN MATHEMATICAL SOCIETY 14 (1999): 65-93.

[GP2] Gallego, Francisco Javier, and B. P. Purnaprajna. "Projective normality and syzygies of algebraic surfaces." Journal für die reine und angewandte Mathematik (Crelles Journal) 1999.506 (1999): 145-180.

[H2009] Hassett, Brendan. "Rational surfaces over nonclosed fields." Arithmetic geometry 8 (2009): 155-209.

[HX2013] Hacon, Christopher D., and Chenyang Xu. "Existence of log canonical closures." Inventiones mathematicae 192.1 (2013): 161-195.

[Liu] Liu, Qing, and Reinie Erne. Algebraic geometry and arithmetic curves. Oxford university press, 2002.

[M] Mumford, David. "Varieties defined by quadratic equations." Questions on algebraic varieties. Springer Berlin Heidelberg, 2011. 29-100.

[P] Purnaprajna, B. P. "Some results on surfaces of general type." Canadian journal of mathematics= Journal canadien de mathématiques 4 (2005): 724-749.

[Tan] Tanaka, Hiromu. "Minimal models and abundance for positive characteristic log surfaces." Nagoya Mathematical Journal (2015).

[Ter] Terakawa, Hiroyuki. "The d-very ampleness on a projective surface in positive characteristic." Pacific J. of Math 187 (1999): 187-198.

[Xu] Xu, Chenyang. "On base point free theorem of threefolds in positive characteristic." arXiv preprint arXiv:1311.3819 (2013).