

# Adjoint Rings on Surfaces over DVRs

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## Abstract

Finite generation of various types of adjoint rings of a surface over a DVR. Updates posted at [divisibility.wordpress.com](http://divisibility.wordpress.com)

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## 1 Intro

Recall the property  $N_p$ :

**Definition 1.** ([EL],  $M_F$  and Property  $N_p$ ) Let  $X$  be a projective variety,  $F$  a globally generated vector bundle on  $X$ . Let  $M_F$  be the kernel

$$0 \rightarrow M_F \rightarrow H^0(F) \otimes \mathcal{O}_X \rightarrow F \rightarrow 0 \quad \star$$

For  $L$  ample and globally generated line bundle,  $L$  satisfies property  $N_p$  if

$$H^1(L^{\otimes r}) = 0$$

for all  $r \geq 1$  and

$$H^1(M_L^{p+1} \otimes L^s) = 0$$

for all  $0 \leq p' \leq p$ .

Tensoring the exact sequence  $\star$  with  $L$ , we have

$$\cdots \rightarrow H^0(L) \otimes H^0(L) \rightarrow H^0(2L) \rightarrow H^1(M_L \otimes L) \rightarrow H^1(L) \otimes H^0(L) \rightarrow \cdots$$

so in order to show property  $N_0$  for  $L = K_X + 4B$ , the second condition can be replaced by showing that the multiplication map

$$H^0(L)^2 \rightarrow H^0(2L) \star \star$$

is surjective. Note that this surjectivity is equivalent to the adjoint ring  $R(L)$  being generated in degree 1.

In characteristic 0, Gallego, Purnaprajna, Hanumanthu, and Banagere (**write all the various references here**) have proved various results along the lines of the following:

**Theorem 2.** [P, 3.14] *Let  $S$  be a surface of general type, let  $A$  be an ample line bundle and let  $m = \left\lceil \frac{(A \cdot (K_S + 4A) + 1)^2}{2A^2} \right\rceil$ . Let  $L = K_S \otimes A^{\otimes n}$ . If  $n \geq 2m$ , then  $L$  satisfies property  $N_0$  and even  $N_1$ .*

The first aim of this article is to prove a theorem similar to the above, but on a surface over a DVR  $R$  in positive or mixed characteristic. This article is concerned with the basic techniques, and presumably various tricks from the characteristic 0 case employed by the aforementioned authors could be used to obtain sharper results. Specifically, one would like to prove the following:

**Theorem 3.** *Let  $X/R$  be a smooth surface of general type, let  $A$  be an ample line bundle, and let  $L = K_X + nA$ . There is a computable constant  $M$  such that if  $n > M$ , then  $L$  satisfies  $N_0$  and  $N_1$ .*

Thus:

**Corollary 4.** *Let  $X/R$  be a smooth surface with  $K_X$  ample, then the canonical ring is generated in degree  $M$  where  $M$  is a computable constant.*

Another consequence of these techniques is the following (explain more)

**Theorem 5.** *Let  $(X, \Delta)$  be a klt pair of relative dimension 2 over a DVR  $R$  with perfect residue field  $k$  and perfect fraction field  $K$ . Assume that  $K_X + \Delta$  is a pseudo-effective  $\mathbb{Q}$ -Cartier divisor, which is simple normal crossings over  $R$ . Then the adjoint ring  $R(K_X + \Delta, K_X + \Delta + A)$  is finitely generated where  $A$  is an ample divisor.*

*Proof.* (Sketch) Let  $m, n$  be integers such that  $m > n + 2 \gg 0$ . C.f. [CZ, 2.11], one must show surjectivity of the following multiplication map

$$H^0(ma(K_X + \Delta + A) + nb(K_X + \Delta) - a(K_X + \Delta + A) = D) \\ \otimes H^0(a(K_X + \Delta + A) = D') \rightarrow H^0(D + D').$$

Choose  $a$  an integer large enough so that [Eg, cor 28. ] applies to the second module in the above product. Note that

$$ma(K_X + \Delta + A) + nb(K_X + \Delta) - a(K_X + \Delta + A) \sim$$

$$((m-1)a + nb)(K_X + \Delta) + (m-1)aA \sim$$

$$((m-1)a + nb)(K_X + \Delta) + ((m-1)a + nb)dA'$$

$d = \frac{(m-1)a}{((m-1)a + nb)}$  so choose  $b$  large enough that [Eg, cor 28.] applies to the first module in the above product. By the Nakayama's lemma surjectivity of the above map can be checked on the special fiber (possibly after base change so the residue field is algebraically closed).

To check surjectivity on the special fiber, we check finite generation of  $R(K_{X_k} + \Delta_k, K_{X_k} + \Delta_k + A_k)$ . [Tan] applies to give termination of the minimal model program (hence the minimal model program with scaling of an ample divisor  $A$  terminates), thus c.f. [HX2013, lemma 2.7] there exists a rational number  $t_0 > 0$  and a birational contraction  $\phi : X_k \rightarrow X'_k$  which is a  $K_{X_k} + \Delta_k + tA_k$  good minimal model for all  $0 \leq t \leq t_0$ , and so, as nef implies semiample by [Tan], then c.f.[H2009, 5.5] gives that semi-ample divisors in arbitrary characteristic have finitely generated Cox ring, the ring  $R(X_k, K_{X_k} + \Delta_k, K_{X_k} + \Delta_k + A_k)$  is finitely generated.  $\square$

As a corollary to the above, c.f. [CL],

**Corollary 6.** *Let  $(X, \Delta)$  be a klt pair of relative dimension 2 over a DVR  $R$  with perfect residue field  $k$  and perfect fraction field  $K$ . Assume that  $K_X + \Delta$  is a pseudo-effective  $\mathbb{Q}$ -Cartier divisor, which is simple normal crossings over  $R$ . Then the minimal model of  $(X, \Delta)$  exists over  $R$ . Also, nefness of  $K_X + \Delta$  implies semi-ampleness.*

*Proof.* 4 (sketch) (Essentially [CL, 6.8]) Suppose  $A$  is a sufficiently general ample  $\mathbb{Q}$ -divisor such that  $(X, \Delta + A)$  is klt and  $K_X + \Delta + A$  is nef. Let  $\{\lambda_i\}$  be a sequence of positive real numbers corresponding to a minimal model program with scaling of  $A = A_1$ . Note that

$$\begin{aligned} R_i &= R(X_i; K_{X_i} + \Delta_i, K_{X_i} + \Delta_i + \lambda_i A_i) \\ &\approx R(X_1; K_{X_1} + \Delta_1, K_{X_1} + \Delta_1 + \lambda_i A_1) \end{aligned}$$

are all finitely generated by Theorem 5. The remainder of the proof is given in section 4.  $\square$

## 2 Background

The following results are necessary.

**Theorem 7.** *Let  $X$  be a smooth projective surface over an algebraically closed field of characteristic  $p > 0$  and let  $D$  be a big and nef Cartier divisor on  $X$ . Assume that either*

- (1)  $\kappa(X) \neq 2$  and  $X$  is not quasi-elliptic with  $\kappa(X) = 1$ ; or
- (2)  $X$  is of general type with  $p \geq 3$  and  $(D^2) > \text{vol}(X)$  or  $p = 2$  and  $(D^2) > \max\{\text{vol}(X), \text{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$ .

Then

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all  $i > 0$ .

**Theorem 8.** [DF, 4.11] (*Effective Base-point Freeness*) *Let  $X$  be a surface of general type and let  $D$  be a big and nef divisor on  $X$ . Then the following holds.*

*If  $p \geq 3$  and  $D^2 > \text{vol}(X) + 4$  and  $|K_X + D|$  has a base point at  $x \in X$ , then there exists a curve  $C$  such that  $D.C \leq 1$ .*

*If  $p = 2$  and if  $D^2 > \text{vol}(X) + 6$  and  $|K_X + D|$  has a base point at  $x \in X$ , then there exists a curve  $C$  such that  $D.C \leq 7$ .*

**Theorem 9.** [DF, 1.2] Let  $D$  and  $B$  be respectively an ample divisor and a nef divisor on a smooth surface  $X$  over an algebraically closed field  $k$ , with  $\text{char } k = p > 0$ . Then  $mD - B$  is very ample for any

$$m > \frac{2D \cdot (H + B)}{D^2} ((K_X + 2D) \cdot D + 1)$$

where

$H = K_X + 4D$  if  $X$  is neither quasi-elliptic with  $\kappa(X) = 1$  nor of general type

$H = K_X + 8D$  if  $X$  is quasi-elliptic with  $\kappa(X) = 1$  and  $p = 3$ ;

$H = K_X + 19D$  if  $X$  is quasi-elliptic with  $\kappa(X) = 1$  and  $p = 2$ ;

$H = 2K_X + 4D$  if  $X$  is of general type and  $p \geq 3$ ;

$H = 2K_X + 19D$  if  $X$  is of general type and  $p = 2$ .

**Theorem 10.** (Nakayama's Lemma) Let  $M_0 = M/\mathfrak{m}M$ , where  $M$  is a module over a dvr  $R$ . If  $\varphi : M \rightarrow N$  is a homomorphism of  $R$ -modules such that  $\varphi_0 : M_0 \rightarrow N_0$  is surjective, then  $\varphi$  is surjective.

The following theorem of Mumford is important to show surjectivity of multiplication maps.

**Theorem 11.** ([M], Theorem 2) Suppose  $L$  is an ample invertible sheaf on a variety  $X$  such that  $\Gamma(L)$  has no base points. Suppose  $\mathcal{F}$  is a coherent sheaf on  $X$  such that

$$H^i(\mathcal{F} \otimes L^{-i}) = 0,$$

$i \geq 1$ . Then

$$H^i(\mathcal{F} \otimes L^j) = 0, \quad i + j \geq 0, \quad i \geq 1$$

and

$$S(\mathcal{F} \otimes L^i, L) = 0, \quad i \geq 0$$

where  $S(A, B)$  stands for the cokernel of the map given by multiplying global sections.

the induction lemma

**Lemma 12.** ([GP], 1.4.1) Let  $E$  and  $L_1, \dots, L_r$  be coherent sheaves on a variety  $X$ . Consider the map

$$H^0(E) \otimes H^0(L_1 \otimes \dots \otimes L_r) \xrightarrow{\psi} H^0(E \otimes L_1 \otimes \dots \otimes L_r)$$

and the maps

$$\begin{aligned} H^0(E) \otimes H^0(L_1) &\xrightarrow{\alpha_1} H^0(E \otimes L_1), \\ H^0(E \otimes L_1) \otimes H^0(L_2) &\xrightarrow{\alpha_2} H^0(E \otimes L_1 \otimes L_2) \\ &\dots \\ H^0(E \otimes L_1 \otimes \dots \otimes L_{r-1}) \otimes H^0(L_r) &\xrightarrow{\alpha_r} H^0(E \otimes L_1 \otimes \dots \otimes L_r). \end{aligned}$$

If  $\alpha_1, \dots, \alpha_r$  are surjective, then  $\psi$  is also surjective.

**Theorem 13.** [CP2014, 1.1] *Let  $X$  be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism  $\pi : X' \rightarrow X$  with the following properties:*

- (i)  $X'$  is everywhere regular
- (ii)  $\pi$  induces an isomorphism  $\pi^{-1}(\text{Reg } X) \approx \text{Reg } X'$
- (iii)  $\pi^{-1}(\text{Sing } X)$  is a strict normal crossings divisor on  $X'$ .

**Definition 14.** (Asymptotic Order of Vanishing) Let  $X$  be a normal projective variety,  $D$  an effective  $\mathbb{R}$  divisor and  $\Gamma$  a geometric valuation of  $X$ . Define  $\sigma_\Gamma(D) = \inf \{ \text{mult}_\Gamma D' \mid D \sim_{\mathbb{R}} D' \geq 0 \} \in \mathbb{R}$ .

### 3 Proof Theorem 3

First I explore the properties  $N_0$  and  $N_1$ .  $R$  is a discrete valuation ring with residue field  $k$  and fraction field  $K$ .

**Lemma 15.** *Let  $X/R$  be a smooth surface of general type,  $A$  an ample  $\mathbb{Z}$ -divisor, and  $R$  has perfect residue field  $k$  and perfect fraction field  $K$ . Suppose that  $L = K_X + A$  such that  $K_X + A$  is nef and big. Then*

$$H^1(L^k) = 0$$

for all  $k$ .

*Proof.* Note that  $(K_X + A)^2 > K_X^2 = \text{vol}(X)$ , so Terakawa's vanishing theorem applies to

$$K_X + \underbrace{(A + (k-1)(K_X + A))}_D$$

on  $X_k$ , and by semicontinuity, the result holds on the generic fiber as well.  $\square$

**Lemma 16.** *Let  $X/R$  be a smooth surface of general type,  $A$  an ample  $\mathbb{Z}$ -divisor, and  $R$  has perfect residue field  $k$  and perfect fraction field  $K$ . Suppose that  $L = K_X + A$  so that  $K_X + A$  is nef and big. Then surjectivity of the map*

$$H^0(L|_{X_k}) \otimes H^0(L^{\otimes m}|_{X_k}) \rightarrow H^0(L^{\otimes 2}|_{X_k})$$

for all  $m$  implies property  $N_0$  for  $L$  on  $X/R$ .

*Proof.* As  $X_k \sim 0$  on  $X/R$ , then there is an exact sequence in cohomology

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(m(K_X + A)) \rightarrow \mathcal{O}_X(m(K_X + X_k + A)) \\ \rightarrow \mathcal{O}_{X_k}(m(K_{X_k} + A_k)) \rightarrow 0. \end{aligned}$$

Taking cohomology and applying lemma 15 gives a surjection:

$$H^0(m(K_X + A)/R) \rightarrow H^0(m(K_{X_k} + A_k))$$

for any  $m$ . Now applying Nakayama's lemma to the diagram

$$\begin{array}{ccc} H^0(m(K_X + A)/R) \otimes H^0(K_X + A/R) & \longrightarrow & H^0((m+1)(K_X + A)/R) \\ \downarrow & & \downarrow \\ H^0(m(K_{X_k} + A_k)) \otimes H^0(K_{X_k} + A_k) & \longrightarrow & H^0((m+1)(K_{X_k} + A_k)) \end{array}$$

gives that surjectivity of the bottom map implies surjectivity of the top map. From the exact sequence:

$$\begin{aligned} \rightarrow H^0(L^{\otimes m}) \otimes H^0(L) \rightarrow H^0(L^{\otimes m+1}) \\ \rightarrow H^1(M_L^1 \otimes L^m) \rightarrow H^1(L^{\otimes m}) \otimes H^0(L) \rightarrow \end{aligned}$$

then applying lemma 15 it suffices to show surjectivity of the top map in the above diagram for each  $m \geq 1$ .  $\square$

**Lemma 17.** *Let  $X/R$  be a smooth surface of general type,  $A$  an ample  $\mathbb{Z}$ -divisor, and  $R$  has perfect residue field  $k$  and perfect fraction field  $K$ . Suppose that  $r > 3$  and define  $L_r = K_{X_k} + rA_k$  so that  $K_{X_k} + A_k$  and  $A_k = lA'_k$  are base-point free and  $A_k - K_{X_k}$  is nef and big with  $(A_k - K_{X_k})^2 > K_{X_k}^2$  (such an  $r$  and  $l$  are computable based on the various intersection numbers of  $K_{X_k}$  and*

$A_k$  over an algebraically closed field of characteristic  $p > 0$  using theorems 8, 9). Then

$$H^0(L_r|_{X_k}) \otimes H^0(L_r^{\otimes m}|_{X_k}) \rightarrow H^0(L_r^{\otimes m+1}|_{X_k})$$

is surjective for all  $m$ .

*Proof.* Suppose first that  $m = 1$ , the other cases will follow similarly. By Lemma 12, it suffices to show that

$$\begin{aligned} H^0(K_{X_k} + rA_k) \otimes H^0(K_{X_k} + A_k) &\rightarrow H^0(2K_{X_k} + (r+1)A_k), \\ H^0(2K_{X_k} + (r+1)A_k) \otimes H^0(A_k) &\rightarrow H^0(2K_{X_k} + (r+2)A_k), \\ &\dots, \\ H^0(2K_{X_k} + (2r-1)A_k) \otimes H^0(A_k) &\rightarrow H^0(2K_{X_k} + 2rA_k) \end{aligned}$$

are surjective. By Lemma 11, the first surjectivity follows from the vanishing of

$$H^1((r-1)A_k) = H^1(K_{X_k} + (r-1)A_k - K_{X_k})$$

and

$$H^2((r-2)A_k - K_{X_k}) = H^2(2K_{X_k} - (r-2)A_k).$$

The first vanishing holds by Terakawa's Theorem 7 since by assumption  $A_k - K_{X_k}$  is nef and big with  $(A_k - K_{X_k})^2 > K_{X_k}^2$ . The second vanishing theorem holds since  $2(A_k - K_{X_k}) \geq 0$  and  $r > 3$ . The other surjectivities follow similarly.

If  $m = 2$ , then similar logic applies to the maps

$$\begin{aligned} H^0(2(K_{X_k} + rA_k)) \otimes H^0(K_{X_k} + A_k) &\rightarrow H^0(3K_{X_k} + (2r+1)A_k), \\ H^0(3K_{X_k} + (2r+1)A_k) \otimes H^0(A_k) &\rightarrow H^0(3K_{X_k} + (2r+2)A_k), \\ &\dots, \\ H^0(3K_{X_k} + (3r-1)A_k) \otimes H^0(A_k) &\rightarrow H^0(3K_{X_k} + 3rA_k) \end{aligned}$$

and the necessary vanishings are slightly easier.  $\square$

The above proposition completes the  $N_0$  portion of Theorem 3.

*Claim 18.*  $N_1$  also holds for  $L_r$  (possibly slightly enlarging  $l$  and  $r$  as defined in the previous lemma).



*Proof.* We must show, in addition to what was proved in the first claim, that  $H^1(M_L^{\otimes 2} \otimes L^{\otimes b}) = 0$  for all  $b \geq 1$ . I'll start with the  $b = 1$  case for simplicity. There is an exact sequence:

$$\begin{aligned} H^0(M_L^{\otimes 2} \otimes L) &\rightarrow H^0(M_L \otimes L) \otimes H^0(L) \\ &\rightarrow H^0(M_L \otimes 2L) \rightarrow H^1(M_L^2 \otimes L) \rightarrow H^1(M_L \otimes L) \otimes H^0(L) \rightarrow \dots \end{aligned}$$

it suffices to show surjectivity of the second map. Using Lemma 12 and Lemma 11 as in the argument for  $N_0$ , it suffices to show (1) that

$$\begin{aligned} H^1(M_L \otimes L \otimes \mathcal{O}_X(-K_X - A)) &= 0, \\ H^1(M_L \otimes L \otimes \mathcal{O}_X(K_X + A - A)) &= 0, \\ &\dots, \\ H^1(M_L \otimes L \otimes \mathcal{O}_X(K_X + (r-1)A - A)) &= 0, \end{aligned}$$

and these follow as in the  $N_0$  case after possibly enlarging  $r, l$ ; and (2) also that

$$\begin{aligned} H^2(M_L \otimes L \otimes \mathcal{O}_X(-K_X - 2A)) &= 0, \\ H^2(M_L \otimes L \otimes \mathcal{O}_X(K_X + A - 2A)) &= 0, \\ &\dots, \\ H^2(M_L \otimes L \otimes \mathcal{O}_X(K_X + (r-1)A - 2A)) &= 0. \end{aligned}$$

Continuing the above exact sequence to second cohomology gives:

$$\begin{aligned} &\rightarrow H^1(M_L \otimes L) \otimes H^0(L) \rightarrow H^1(M_L \otimes L^{\otimes 2}) \\ &\rightarrow H^2(M_L^{\otimes 2} \otimes L) \rightarrow H^2(M_L \otimes L) \otimes H^0(L) \rightarrow \end{aligned}$$

so it suffices to show both  $H^1(M_L \otimes L^{\otimes 2}) = 0$  and  $H^2(M_L \otimes L) = 0$ . Vanishing of  $H^1(M_L \otimes L^{\otimes 2})$  follows from property  $N_0$  and for the second vanishing, consider

$$\rightarrow H^1(L) \otimes H^0(L) \rightarrow H^1(L^{\otimes 2}) \rightarrow H^2(M_L \otimes L) \rightarrow H^2(L) \otimes H^0(L) \rightarrow .$$

Since  $H^1(L^{\otimes 2}) = 0$  by Lemma 15, and then it suffices to show vanishing of  $H^2(K_{X_k} + rA_k) \approx H^0(K_{X_k} - (K_{X_k} - rA_k))$ . This is clear. For  $b > 1$ , the above arguments are similar (easier since  $L^{\otimes 2}$  is more positive than just  $L$ ).  $\square$

## 4 Sketch of Corollary 6

**Lemma 19.** (*[CL, 3.5], [CL, 3.9]*) *The cited Theorems holds in the case of arbitrary field.*

*Proof.* (tbd) c.f. Various references on Cox Ring over fields, e.g. [H2009].  $\square$

**Theorem 20.** *Let  $(X, \Delta)$  be a projective  $\mathbb{Q}$ -factorial klt pair of relative dimension 2, log smooth over a DVR  $R$  with algebraically closed residue field  $k$ . Let  $A$  be sufficiently general ample  $\mathbb{Q}$ -divisor on  $X$  such that  $(X, \Delta + A)$  is klt and  $K_X + \Delta + A$  is nef. The minimal model program with scaling of  $A$  terminates.*

The proof will be given in two reductions. The first reduction is to the pseudo-effective case. The second reduction is from the pseudo-effective case to the big case. Success is declared when there is a morphism  $f : (X, \Delta) \rightarrow (X_n, \Delta_n)$  to a pair over a DVR such that on both geometric fibers  $K_{X_n} + \Delta_n$  is nef.

*Proof.* (Essentially just a slightly modified version of [CL, 6.5,6.8]) First suppose that  $K_X + \Delta$  is pseudo-effective.

Consider a minimal model program with scaling of  $A_1 = A$  run (slightly different from usual) as follows. Let

$$\lambda = \inf \{t \geq 0 \mid K_X + \Delta + tA_1 \text{ is nef on both fibers}\}.$$

If  $\lambda = 0$ , then we are done. Otherwise, there exists a  $(K_X + \Delta)$ -negative extremal ray  $C$  on one or both fibers such that  $(K_X + \Delta + \lambda A) \cdot C \geq 0$  on both fibers and is 0 on one of them. Take the contraction on  $(X, \Delta) / R$  given by Theorem 13 and replace  $(X, \Delta) / R = (X_1, \Delta_1)$  by the corresponding flip or contraction  $\phi : (X_1, \Delta_1) \rightarrow (X_2, \Delta_2) / R$ . Let  $A_2 = \phi_* A_1$  and  $\Delta_2 = \phi_* \Delta_1$ . Then  $K_{X_2} + \Delta_2 + \lambda A_2$  is nef on both fibers, so we repeat the process.

As in [CL, 6.8], consider

$$\begin{aligned} R_i &= R(X_i, K_{X_i} + \Delta_i, K_{X_i} + \Delta_i + \lambda_i A_i) \\ &\approx R(X_1, K_{X_1} + \Delta_1, K_{X_1} + \Delta_1 + \lambda_i A_1) \end{aligned}$$

which are finitely generated by Theorem 5. Lemma 19 implies that

$$\text{Supp } R_{i_0, k} \text{ or } K$$

$$\begin{aligned}
&= \mathbb{R}_+ (K_{X_{i_0}} + \Delta_{i_0}) + \mathbb{R}_+ (K_{X_{i_0}} + \Delta_{i_0} + \lambda_{i_0} A_{i_0}) \\
&= \bigcup \mathcal{C}_j^{i_0}
\end{aligned}$$

a rational polyhedral subdivision on both fibers. Let  $C_{j,k}^i, C_{j,K}^i$  denote the proper transform of  $C_{j,k}^{i_0}$  and  $C_{j,K}^{i_0}$  respectively if  $i \geq i_0$ . By Lemma 19, for each geometric valuation  $\sigma_{\Gamma,k}, \sigma_{\Gamma,K}$  is linear on  $C_{j,k}^{i_0}$  and  $C_{j,K}^{i_0}$  respectively.

Note it is impossible that  $D_i = K_{X_i} + \Delta_i + \lambda_i A_i \in \text{int } \mathcal{C}_j^i$  on both fibers at once: By non-vanishing,  $D_i$  is semiample over  $R$ , and  $\sigma_{\Gamma,k}, \sigma_{\Gamma,K}$  are linear and non-negative on  $C_{j,k}^i, C_{j,K}^i$  thus identically zero, thus by finite generation on each fiber (c.f. [Tan]) any divisor in  $C_{j,k}^i, C_{j,K}^i$  are nef, so  $\lambda_i$  is not the smallest such that  $K_{X_i} + \Delta_i + \lambda_i A_i$  is nef on both fibers simultaneously, contradicting the construction. As there are only finitely many boundary rays in both  $C_{j,k}^{i_0}$  and  $C_{j,K}^{i_0}$  together, and  $K_{X_i} + \Delta_i + \lambda_i A_i$  must be at a boundary ray on one of the fibers, the set  $\{\lambda_i\}$  is finite. Letting  $\lambda$  be the minimum of the  $\lambda_i$ , if  $\lambda \neq 0$ , then it suffices to consider the minimal model program with scaling of  $(1 - \epsilon) A_i = A'_i$  for  $K_{X_{i_0}} + \Delta_{i_0} + \epsilon A_{i_0}$  for  $0 < \epsilon < \lambda$ .

Let  $\{\lambda'_i\}$  be a sequence of numbers corresponding to the second part of the minimal model program with scaling of  $A'_i$ . Note that  $K_{X_i} + \Delta_i + \epsilon A_i \in \text{int } \text{supp } R_{i,k} \text{ or } K$ , for all  $i > i_0$  so  $K_{X_i} + \Delta_i + \lambda_i A_i \in \text{int } \text{supp } R_{i,k} \text{ or } K$ , for each  $i$ .  $K_{X_i} + \Delta_i$ , note as  $\text{supp } R_{i,k} \text{ or } K$ , contains an open neighborhood of nef divisors, it contains ample divisors in its interior. Applying Lemma 19,  $\varphi(C_j^1)$  is contained in the pullback of the nef cone on  $X_{i,k} \text{ or } K$  for some  $j$ . Thus there are either finitely many  $X_i, k \text{ and } K$  (so we are done) or at some point, the pullbacks of the nef cones for different  $X_i, X_{i' \neq i}$  on either (perhaps not simultaneously, but eventually it must be so) have a common interior point. Applying negativity lemma, which holds on either fiber, [Xu, 2.1], gives a morphism between  $X_i \rightarrow X_{i'}$ .  $\square$

## References

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