Adjoint Rings on Surfaces over DVRs 0.2

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Abstract

Finite generation of various types of adjoint rings of a surface over a DVR. Updates posted at divisibility.wordpress.com

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1 Intro

Recall the property N_p:

Definition 1. ([EL], M_F and Property N_p) Let X be a projective variety, F a globally generated vector bundle on X. Let M_F be the kernel

$$0 \to M_F \to H^0(F) \otimes \mathcal{O}_X \to F \to 0 \quad \star$$

For L ample and globally generated line bundle, L satisfies property N_p if

$$H^1\left(L^{\otimes r}\right) = 0$$

for all $r \geq 1$ and

$$H^1\left(M_L^{p+1} \otimes L^s\right) = 0$$

for all $0 \leq p' \leq p$.

Tensoring the exact sequence $\star$ with L, we have

$$\cdots \to H^0(L) \otimes H^0(L) \to H^0(2L) \to H^1(M_L \otimes L) \to H^1(L) \otimes H^0(L) \to \cdots$$

so in order to show property N_0 for $L = K_X + 4B$, the second condition can be replaced by showing that the multiplication map

$$H^0(L)^2 \to H^0(2L) \ \star\star$$

is surjective. Note that this surjectivity is equivalent to the adjoint ring $R(L)$ being generated in degree 1.

In characteristic 0, Gallego, Purnaprajna, Hanumanthu, and Banagere**(write all the various references here)** have proved various results along the lines of the following:

Theorem 2. *[P, 3.14] Let S be a surface of general type, let A be an ample line bundle and let $m = \left[\frac{(A.(K_S+4A)+1)^2}{2A^2}\right]$. Let $L = K_S \otimes A^{\otimes n}$. If $n \geq 2m$, then L satisfies property N_0 and even N_1.*

The first aim of this article is to prove a theorem similar to the above, but on a surface over a DVR R in positive or mixed characteristic. This article is concerned with the basic techniques, and presumably various tricks from the characteristic 0 case employed by the aforementioned authors could be used to obtain sharper results. Specifically, one would like to prove the following:

Theorem 3. *Let X/R be a smooth surface of general type, let A be an ample line bundle, and let $L = K_X + nA$. There is a computable constant M such that if $n > M$, then L satisfies N_0 and N_1.*

Thus:

Corollary 4. *Let X/R be a smooth surface with K_X ample, then the canonical ring is generated in degree M where M is a computable constant.*

Another consequence of these techniques is the following (explain more)

Theorem 5. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k and perfect fraction field K. Assume that $K_X + \Delta$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over R. Then the adjoint ring $R(K_X + \Delta, K_X + \Delta + A)$ is finitely generated where A is an ample divisor.*

Proof. (Sketch) Let m, n be integers such that $m > n + 2 \gg 0$. C.f. [CZ, 2.11], one must show surjectivity of the following multiplication map

$$H^0 \left(ma \left(K_X + \Delta + A\right) + nb \left(K_X + \Delta\right) - a \left(K_X + \Delta + A\right) = D\right) \otimes H^0 \left(a \left(K_X + \Delta + A\right) = D'\right) \rightarrow$$

Choose a an integer large enough so that [Eg, cor 28.] applies to the second module in the above product. Note that

$$ma \left(K_X + \Delta + A\right) + nb \left(K_X + \Delta\right) - a \left(K_X + \Delta + A\right) \sim$$

$$\left(\left(m-1\right)a + nb\right)\left(K_X + \Delta\right) + \left(m-1\right)aA \sim$$
$$\left(\left(m-1\right)a + nb\right)\left(K_X + \Delta\right) + \left(\left(m-1\right)a + nb\right)dA'$$

$d = \frac{(m-1)a}{((m-1)a+nb)}$ so choose b large enough that [Eg, cor 28.] applies to the first module in the above product. By the Nakayama's lemma surjectivity of the above map can be checked on the special fiber (possibly after base change so the residue field is algebraically closed).

To check surjectivity on the special fiber, we check finite generation of $R\left(K_{X_k} + \Delta_k, K_{X_k} + \Delta_k + A_k\right)$. [Tan] applies to give termination of the minimal model program (hence the minimal model program with scaling of an ample divisor A terminates), thus c.f. [HX2013, lemma 2.7] there exists a rational number $t_0 > 0$ and a birational contraction $\phi : X_k \rightarrow X'_k$ which is a $K_{X_k} + \Delta_k + tA_k$ good minimal model for all $0 \leq t \leq t_0$, and so, as nef implies semiample by [Tan], then c.f.[H2009, 5.5] gives that semi-ample divisors in arbitrary characteristic have finitely generated Cox ring, the ring $R\left(X_k, K_{X_k} + \Delta_k, K_{X_k} + \Delta_k + A_k\right)$ is finitely generated. □

As a corollary to the above, c.f. [CL],

Corollary 6. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k and perfect fraction field K. Assume that $K_X + \Delta$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over R. Then the minimal model of (X, Δ) exists over R. Also, nefness of $K_X + \Delta$ implies semi-ampleness.*

Proof. 4 (sketch) (Essentially [CL, 6.8]) Suppose A is a sufficiently general ample $\mathbb{Q}$-divisor such that $(X, \Delta + A)$ is klt and $K_X + \Delta + A$ is nef. Let $\{\lambda_i\}$

be a sequence of positive real numbers corresponding to a minimal model program with scaling of $A = A_1$. Note that

$$R_i = R(X_i; K_{X_i} + \Delta_i, K_{X_i} + \Delta_i + \lambda_i A_i)$$
$$\approx R(X_1; K_{X_1} + \Delta_1, K_{X_1} + \Delta_1 + \lambda_i A_1)$$

are all finitely generated by Theorem 5. The remainder of the proof is given in section 4. □

2 Background

The following results are necessary.

Theorem 7. *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X. Assume that either*
(1) $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or
(2) X is of general type with
$p \geq 3$ and $(D^2) > vol(X)$ or
$p = 2$ and $(D^2) > max\{vol(X), vol(X) - 3\chi(\mathcal{O}_X) + 2\}$.
Then

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all $i > 0$.

Theorem 8. *[DF, 4.11] (Effective Base-point Freeness) Let X be a surface of general type and let D be a big and nef divisor on X. Then the following holds.*

If $p \geq 3$ and $D^2 > vol(X) + 4$ and $|K_X + D|$ has a base point at $x \in X$, then there exists a curve C such that $D.C \leq 1$.

If $p = 2$ and if $D^2 > vol(X) + 6$ and $|K_X + D|$ has a base point at $x \in X$, then there exists a curve C such that $D.C \leq 7$.

Theorem 9. *[DF, 1.2] Let D and B be respectively an ample divisor and a nef divisor on a smooth surface X over an algebraically closed field k, with char $k = p > 0$. Then $mD - B$ is very ample for any*

$$m > \frac{2D.(H + B)}{D^2}((K_X + 2D).D + 1)$$

where

$H = K_X + 4D$ *if* X *is neither quasi-elliptic with* $\kappa(X) = 1$ *nor of general type*

$H = K_X + 8D$ *if* X *is quasi-elliptic with* $\kappa(X) = 1$ *and and* $p = 3$*;*

$H = K_X + 19D$ *if* X *is quasi-elliptic with* $\kappa(X) = 1$ *and* $p = 2$*;*

$H = 2K_X + 4D$ *if* X *is of general type and* $p \geq 3$*;*

$H = 2K_X + 19D$ *if* X *is of general type and* $p = 2$*.*

Theorem 10. *(Nakayama's Lemma) Let* $M_0 = M/\mathfrak{m}M$ *, where* M *is a module over a dvr* R*. If* $\varphi : M \to N$ *is a homomorphism of* R*-modules such that* $\varphi_0 : M_0 \to N_0$ *is surjective, then* φ *is surjective.*

The following theorem of Mumford is important to show surjectivity of multiplication maps.

Theorem 11. *([M], Theorem 2) Suppose* L *is an ample invertible sheaf on a variety* X *such that* $\Gamma(L)$ *has no base points. Suppose* $\mathcal{F}$ *is a coherent sheaf on* X *such that*

$$H^i\left(\mathcal{F} \otimes L^{-i}\right) = 0,$$

$i \geq 1$*. Then*

$$H^i\left(\mathcal{F} \otimes L^j\right) = 0,\ i + j \geq 0,\ i \geq 1$$

and

$$S\left(\mathcal{F} \otimes L^i, L\right) = 0,\ i \geq 0$$

where $S(A, B)$ *stands for the cokernel of the map given by multiplying global sections.*

the induction lemma

Lemma 12. *([GP], 1.4.1) Let* E *and* $L_1, ..., L_r$ *be coherent sheaves on a variety* X*. Consider the map*

$$H^0(E) \otimes H^0(L_1 \otimes \cdots \otimes L_r) \stackrel{\psi}{\to} H^0(E \otimes L_1 \otimes \cdots \otimes L_r)$$

and the maps

$$H^0(E) \otimes H^0(L_1) \stackrel{\alpha_1}{\to} H^0(E \otimes L_1),$$
$$H^0(E \otimes L_1) \otimes H^0(L_2) \stackrel{\alpha_2}{\to} H^0(E \otimes L_1 \otimes L_2)$$

$$\cdots$$

$$H^0\left(E\otimes L_1\otimes\cdots\otimes L_{r-1}\right)\otimes H^0\left(L_r\right)\stackrel{\alpha_r}{\to}H^0\left(E\otimes L_1\otimes\cdots\otimes L_r\right).$$

If $\alpha_1,...,\alpha_r$ are surjective, then ψ is also surjective.

Theorem 13. *[CP2014, 1.1] Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*
(i) X' is everywhere regular
(ii) π induces an isomorphism $\pi^{-1}(Reg\ X) \approx Reg\ X$
(iii) $\pi^{-1}(Sing\ X)$ is a strict normal crossings divisor on X'.

3 Proof Theorem 3

First I explore the properties N_0 and N_1.

Lemma 14. *Let X/R be a smooth surface of general type, A an ample $\mathbb{Z}$-divisor, and R has perfect residue field k and perfect fraction field K. Suppose that $L = K_X + A$ such that $K_X + A$ is nef and big. Then*

$$H^1\left(L^k\right)=0$$

for all k.

Proof. Note that $(K_X+A)^2 > K_X^2 = vol(X)$, so Terakawa's vanishing theorem applies to

$$K_X+\underbrace{\left(A+(k-1)\left(K_X+A\right)\right)}_{D}$$

on X_k, and by semicontinuity, the result holds on the generic fiber as well. □

Lemma 15. *Let X/R be a smooth surface of general type, A an ample $\mathbb{Z}$-divisor, and R has perfect residue field k and perfect fraction field K. Suppose that $L = K_X + A$ so that $K_X + A$ is nef and big. Then surjectivity of the map*

$$H^0\left(L|_{X_k}\right)\otimes H^0\left(L^{\otimes m}|_{X_k}\right)\to H^0\left(L|_{X_k}^{\otimes 2}\right)$$

for all m implies property N_0 for L on X/R.

Proof. As $X_k \sim 0$ on X/R, then there is an exact sequence in cohomology

$$0 \to \mathcal{O}_X\left(m\left(K_X + A\right)\right) \to \mathcal{O}_X\left(m\left(K_X + X_k + A\right)\right) \to \mathcal{O}_{X_k}\left(m\left(K_{X_k} + A_k\right)\right) \to 0.$$

Taking cohomology and appliying lemma 14 gives a surjection:

$$H^0\left(m\left(K_X + A\right)/R\right) \to H^0\left(m\left(K_{X_k} + A_k\right)\right)$$

for any m. Now applying Nakayama's lemma to the diagram

$$\begin{array}{ccc} H^0\left(m\left(K_X + A\right)/R\right) \otimes H^0\left(K_X + A/R\right) & \longrightarrow & H^0\left(\left(m+1\right)\left(K_X + A\right)/R\right) \\ \downarrow & & \downarrow \\ H^0\left(m\left(K_{X_k} + A_k\right)\right) \otimes H^0\left(K_{X_k} + A_k\right) & \longrightarrow & H^0\left(\left(m+1\right)\left(K_{X_k} + A_k\right)\right) \end{array}$$

gives that surjectivity of the bottom map implies surjectivity of the top map. From the exact sequence:

$$\to H^0\left(L^{\otimes m}\right) \otimes H^0\left(L\right) \to H^0\left(L^{\otimes m+1}\right) \to H^1\left(M_L^1 \otimes L^m\right) \to H^1\left(L^{\otimes m}\right) \otimes H^0\left(L\right) \to$$

then applying lemma 14 it suffices to show surjectivity of the top map in the above diagram for each $m \geq 1$. □

Lemma 16. *Let X/R be a smooth surface of general type, A an ample $\mathbb{Z}$-divisor, and R has perfect residue field k and perfect fraction field K. Suppose that $r > 3$ and define $L_r = K_{X_k} + rA_k$ so that $K_{X_k} + A_k$ and $A_k = lA'_k$ are base-point free and $A_k - K_X$ is nef and big with $\left(A_k - K_{X_k}\right)^2 > K_{X_k}^2$ (such an r and l are computable based on the various intersection numbers of K_{X_k} and A_k over an algebraically closed field of characteristic $p > 0$ using theorems 8, 9). Then*

$$H^0\left(L_r|_{X_k}\right) \otimes H^0\left(L_r^{\otimes m}|_{X_k}\right) \to H^0\left(L_r^{\otimes m+1}|_{X_k}\right)$$

is surjective for all m.

Proof. Suppose first that $m = 1$, the other cases will follow similarly. By Lemma 12, it suffices to show that

$$H^0\left(K_{X_k} + rA_k\right) \otimes H^0\left(K_{X_k} + A_k\right) \to H^0\left(2K_{X_k} + \left(r+1\right)A_k\right),$$

$$H^0\left(2K_{X_k}+(r+1)A_k\right)\otimes H^0\left(A_k\right)\to H^0\left(2K_{X_k}+(r+2)A_k\right),$$
$$\cdots,$$
$$H^0\left(2K_{X_k}+(2r-1)A_k\right)\otimes H^0\left(A_k\right)\to H^0\left(2K_{X_k}+2rA_k\right)$$

are surjective. By Lemma 11, the first surjectivity follows from the vanishing of

$$H^1\left((r-1)A_k\right)=H^1\left(K_{X_k}+(r-1)A_k-K_{X_k}\right)$$

and

$$H^2\left((r-2)A_k-K_{X_k}\right)=H^0\left(2K_{X_k}-(r-2)A_k\right).$$

The first vanishing holds by Terakawa's Theorem 7 since by assumption $A_k-K_{X_k}$ is nef and big with $\left(A_k-K_{X_k}\right)^2>K_{X_k}^2$. The second vanishing theorem holds since $2\left(A_k-K_{X_k}\right)\geq 0$ and $r>3$. The other surjectivities follow similarly.

If $m=2$, then similar logic applies to the maps

$$H^0\left(2\left(K_{X_k}+rA_k\right)\right)\otimes H^0\left(K_{X_k}+A_k\right)\to H^0\left(3K_{X_k}+(2r+1)A_k\right),$$
$$H^0\left(3K_{X_k}+(2r+1)A_k\right)\otimes H^0\left(A_k\right)\to H^0\left(3K_{X_k}+(2r+2)A_k\right),$$
$$\cdots,$$
$$H^0\left(3K_{X_k}+(3r-1)A_k\right)\otimes H^0\left(A_k\right)\to H^0\left(3K_{X_k}+3rA_k\right)$$

and the necessary vanishings are slightly easier. □

The above proposition completes the N_0 portion of Theorem 3.

Claim 17. N_1 also holds for L_r (possibly slightly enlarging l and r as defined in the previous lemma).

Proof. We must show, in addition to what was proved in the first claim, that $H^1\left(M_L^{\otimes 2}\otimes L^{\otimes b}\right)=0$ for all $b\geq 1$. I'll start with the $b=1$ case for simplicity. There is an exact sequence:

$$H^0\left(M_L^{\otimes 2}\otimes L\right)\to H^0\left(M_L\otimes L\right)\otimes H^0\left(L\right)\to H^0\left(M_L\otimes 2L\right)\to H^1\left(M_L^2\otimes L\right)\to H^1\left(M_L\otimes L\right)\otimes H$$

it suffices to show surjectivity of the second map. Using Lemma 12 and Lemma 11 as in the argument for N_0, it suffices to show (1) that

$$H^1\left(M_L\otimes L\otimes\mathcal{O}_X\left(-K_X-A\right)\right)=0,$$

$$H^1\left(M_L \otimes L \otimes \mathcal{O}_X\left(K_X + A - A\right)\right) = 0,$$
$$\cdots,$$
$$H^1\left(M_L \otimes L \otimes \mathcal{O}_X\left(K_X + (r-1)\, A - A\right)\right) = 0,$$

and these follow as in the N_0 case after possibly enlarging r, l; and (2) also that

$$H^2\left(M_L \otimes L \otimes \mathcal{O}_X\left(-K_X - 2A\right)\right) = 0,$$
$$H^2\left(M_L \otimes L \otimes \mathcal{O}_X\left(K_X + A - 2A\right)\right) = 0,$$
$$\cdots,$$
$$H^2\left(M_L \otimes L \otimes \mathcal{O}_X\left(K_X + (r-1)\, A - 2A\right)\right) = 0.$$

Continuing the above exact sequence to second cohomology gives:

$$\to H^1\left(M_L \otimes L\right) \otimes H^0\left(L\right) \to H^1\left(M_L \otimes L^{\otimes 2}\right) \to H^2\left(M_L^{\otimes 2} \otimes L\right) \to H^2\left(M_L \otimes L\right) \otimes H^0\left(L\right) \to$$

so it suffices to show both $H^1\left(M_L \otimes L^{\otimes 2}\right) = 0$ and $H^2\left(M_L \otimes L\right) = 0$. Vanishing of $H^1\left(M_L \otimes L^{\otimes 2}\right)$ follows from property N_0 and for the second vanishing, consider

$$\to H^1\left(L\right) \otimes H^0\left(L\right) \to H^1\left(L^{\otimes 2}\right) \to H^2\left(M_L \otimes L\right) \to H^2\left(L\right) \otimes H^0\left(L\right) \to .$$

Since $H^1\left(L^{\otimes 2}\right) = 0$ by Lemma 14, and then it suffices to show vanishing of $H^2\left(K_{X_k} + rA_k\right) \approx H^0\left(K_{X_k} - \left(K_{X_k} - rA_k\right)\right)$. This is clear. For $b > 1$, the above arguments are similar (easier since $L^{\otimes 2}$ is more positive than just L). □

4 Sketch of Corollary 6

Will be upgraded from "sketch" to "proof" once all the bits are in place

Definition 18. Let X a normal $\mathbb{Q}$-factorial projective variety over k a field of arbitrary characteristic such that $Pic\left(X\right)$ is finitely generated and $Pic\left(X\right)_{\mathbb{Q}} \approx N^1\left(X\right)_{\mathbb{Q}}$. $\Gamma \subset Div\left(X\right)$ a finitely generated group of Cartier divisors on X such that $\Gamma_{\mathbb{Q}} \to Pic\left(X\right)_{\mathbb{Q}} : D \mapsto \mathcal{O}_X\left(D\right)$ is isomorphic. For such a group Γ, the multi-section ring $R_X\left(\Gamma\right) = \bigoplus_{D \in \Gamma} H^0\left(X, \mathcal{O}_X\left(D\right)\right)$ is the Cox ring of X.

Remark 19. [G] If the Cox ring is finitely generated, the cone of effective divisors is rational polyhedral.

Recall we are the situation of $\{\lambda_i\}$ be a sequence of positive real numbers corresponding to a minimal model program with scaling of $A = A_1$ and

$$R_i = R\left(X_i; K_{X_i} + \Delta_i, K_{X_i} + \Delta_i + \lambda_i A_i\right)$$

$$\approx R\left(X_1; K_{X_1} + \Delta_1, K_{X_1} + \Delta_1 + \lambda_i A_1\right) \qquad \star$$

are all finitely generated by Theorem 5.

In order for a sequence of flips to terminate, then c.f. [CL, 6.5,6.8]we need the following to hold:

- [CL] 3.5(1-4), 3.6 (by remark 19) ,
- [CL]3.7(2), 3.8 hold by Theorem 5, and using the identification $\star$ to a log-smooth case.
- [CL, 3.9] by above two points.
- [CL]5.1, 5.2 (characteristic free proofs at least in lower dimensions where we have the resolution of singularities Theorem 13)
- [CL, 5.3, 5.4] hold c.f. [Xu, 2.1]
- [CL, 6.2] holds by Cone Theorem on either fibers, c.f. [Tan].
- [CL, 6.4] clear.

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