# INVARIANCE OF PLURIGENERA AND FINITE GENERATION FOR LOG SURFACES IN MIXED CHARACTERISTIC

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ABSTRACT. Log deformation invariance of plurigenera, and finite generation of the canonical ring, and existence of minimal models for a family of relative dimension 2 over a DVR in mixed characteristic. updates available at divisibility.wordpress.com.

CONTENTS



## 1. INTRODUCTION

The purpose of this article is to consider the mixed characteristic analogue of the famous theorem of Siu in the case of an algebraic variety

$X/R$ of relative dimension 2 over a DVR (and consequences). Siu's theorem states

**Theorem 1.** [Siu] *Let $\pi : X \to \Delta$ be a smooth projective family of compact complex manifolds parametrized by the open unit 1-disk $\Delta$. Assume that the family $\pi : X \to \Delta$ is of general type. Then for every positive integer $m$, the plurigenus $dim_{\mathbb{C}}\Gamma\left(X_t, mK_{X_t}\right)$ is independent of $t \in \Delta$, where $X_t = \pi^{-1}(t)$ and $K_{X_t}$ is the canonical line bundle of $X_t$.*

The above theorem in characteristic 0 has been generalized to all Kodaira dimensions and to Kawamata log-terminal, log smooth pairs c.f. [Siu2], [HMX2010, 1.8], and [BP].

In positive characteristic, the above theorem in general does not hold even when the fibers are surfaces with Kodaira dimension 1, since there are examples of Katsura and Ueno [KU] where a fiber with wild ramification causes the geometric genus to jump rather than being constant in the family. Similarly, Suh [Suh] has constructed counterexamples in Kodaira dimension 2 to invariance of the geometric genus.

However, in their paper, Katsura and Ueno also show that in the case of a smooth algebraic variety $X/R$ over a DVR of relative dimension 2 with residue field $k$ and fraction field $K$, then $\kappa\left(X_k\right) = \kappa\left(X_K\right)$. So the question becomes whether some asymptotic version of Siu's theorem holds in this case. The question this article seeks to answer is whether for $m \gg 0$, it holds that $dim_k\Gamma\left(X_k, mK_{X_k}\right) = dim_K\Gamma\left(X_K, mK_{X_K}\right)$, and if more generally, the same holds for Kawamata log-terminal pairs. As far as the author is aware, the only existing results in this direction are the following result due to Junescue Suh, which uses the techniques of [KU], as well as a $W_2$-lifting hypothesis in place of the Kawamata-Viehweg Vanishing theorem for characteristic 0, and a result due to Tanaka, which assumes a certain ample divisor is added to the pair. Suh's theorem is:

**Theorem 2.** [Suh, 1.2.1(ii), 1.2.4] *Let $R$ be a discrete valuation ring whose fraction field $K$ (resp. residue field $k$) has characteristic zero (resp. is perfect of characteristic $p > 0$) and let $X/R$ be a proper smooth algebraic space of relative dimension 2. If $X_k$ lifts to $W_2(k)$ and is of general type, then one has*

$$P_m(X_K) = P_m(X_k)$$

*for every integer $m \geq 2$. If moreover $X_k$ has reduced picard scheme then $P_m(X_K) = P_m(X_k)$ for all $m \geq 1$.*

Tanaka's theorem is:

**Theorem 3.** [Tan3, 7.3] *Let $X$ be a smooth projective threefold. Let $S$ be a smooth prime divisor on $X$ and let $A$ be an ample $\mathbb{Z}$-divisor on $X$ such that*

*(1) $K_X + S + A$ is nef, and*

*(2) $\kappa(S, K_S + A|_S) \neq 0$.*

*Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that, for every integer $m \geq m_0$, the natural restriction map*

$$H^0(X, m(K_X + S + A)) \to H^0(S, m(K_S + A|_S))$$

*is surjective.*

The proof of the above theorem uses some interesting trace of Frobenius methods which are quite different from the techniques used in this article. In this article, minimal model techniques, combined with [KU] are used to gain a result similar to the above theorems but without the $W_2(k)$ lifting hypothesis or the ample $\mathbb{Z}$-divisor $A$, and with the added benefit that it holds for Kawamata log-terminal pairs. The main theorem is the following.

**Theorem 4.** *Let $(X, \Delta)$ be a kawamata log-terminal pair of relative dimension 2 over a DVR $R$ with perfect residue field $k$ of characteristic $p > 2$ and perfect fraction field $K$. Assume that $(X, \Delta)$ is pseudo-effective, $\mathbb{Q}$-Cartier, and simple normal crossings over $R$. If $\nu(K_X + \Delta) \neq 1$ then, there exists an $m_0$ such that for all $m \in \mathbb{Z}^+$ with $m_0 | m$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

*If $\nu(K_X + \Delta) = 1$ then the same holds if Serre Duality holds on $X$.*

Consequences of the above theorems are finite generation of the canonical ring, the ability to run a minimal model program in mixed characteristic over a DVR in relative dimension 2, as well as the invariance of the log Kodaira dimensions of Kawamata log-terminal pairs in this setting.

**Proposition 5.** *Let $(X, \Delta)$ be a big klt pair of relative dimension 2 over a DVR $R$ with perfect residue field $k$ and perfect fraction field $K$ of characteristic 0. Assume that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over $R$. Then the canonical ring*

*Proj* $R(K_X + \Delta)$ *is finitely generated and a contraction of a* $K_{X_k} + \Delta_k$*-negative extremal ray cut out by an ample* $H$ *on* $R$ *induces a contraction of a* $K_X + \Delta$ *negative extremal ray.*

(Actually, as a result of the following, the big assumption can be removed, although it must be proven first).

**Corollary 6.** *Let* $(X, \Delta)$ *be a klt pair of relative dimension* $2$ *over a DVR* $R$ *with perfect residue field* $k$ *of characteristic* $p > 0$ *and perfect fraction field* $K$*. Assume that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier, pseudo-effective, and simple normal crossings over* $R$*. Then the minimal model of* $(X, \Delta)$ *exists.*

**Corollary 7.** *Let* $(X, \Delta)$ *be a klt pair of relative dimension* $2$ *over a DVR* $R$ *with residue field* $k$ *and fraction field* $K$ *such that* $k$ *is algebraically closed of characteristic* $p > 0$*. Assume* $K_X + \Delta$ *is pseudo-effective and simple normal crossings over* $R$*. Then the numerical Kodaira dimension satisfy*

$$\nu(K_{X_K} + \Delta_K) = \nu(K_{X_k} + \Delta_k).$$

*As a result of abundance for log surfaces, the log Kodaira dimension also satisfy*

$$\kappa(K_{X_K} + \Delta_K) = \kappa(K_{X_k} + \Delta_k).$$

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## 2. Background

Following the notation of [Suh], let $p$ be a prime number and let $R$ be a discrete valuation ring with residue field $k$ and fraction field $K$, such that $K$ is either of characteristic 0 or $p$. For a scheme $Z$ over $R$, $Z_k$ will denote the special fiber $Z \otimes_R k$ and $Z_K$ will denote the generic fiber $Z \otimes_R K$. First a few facts about discrete valuation rings which will help to run the minimal model program over a discrete valuation ring.

The following lemma of Katsura and Ueno is a main ingredient to theorem 4. The proof is included almost verbatim except for noting that the residue field is unchanged after the extension in the proof.

**Lemma 8.** [KU, 9.4] *Let $\varphi : X \to Spec(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $spec(R)$, where $R$ is a DVR with algebraically closed residue field $k$, and field of fractions $K$. Let $X_K$ (resp. $X_k$) denote the generic geometric (resp. closed) fibre of $\varphi$. If $X_k$ contains an exceptional curve of the first kind $e$, there exists a DVR $\tilde{R} \supset R$, with residue field isomorphic to $k$, and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \to \tilde{Y}$ over $Spec\left(\tilde{R}\right)$ such that on the closed fibre, $\pi$ induces the contraction of the exceptional curve $e$. Moreover, on the generic fibre, $\pi$ also induces a contraction of an exceptional curve of the first kind.*

*Proof.* By [A1, Cor 6.2] $Hilb_{X/Spec(R)}$ is represented by an algebraic space $\mathscr{H}$ which is locally of finite type over $Spec(R)$. Let $Y$ be the irreducible component containing the point $\{e\}$ corresponding to the exceptional curve $e$ on the special fiber. Then $e \approx \mathbb{P}^1_k$ and $N_{e/X} \approx \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, so $Y$ is regular at $\{e\}$ and of dimension 1. Since $e$ is fixed in the special fiber, the structure morphism $p : Y \to Spec(R)$ is surjective. By lemma 52 and c.f. remark 53, we can find an étale cover $\tilde{R} \supset R$ and a morphism $j : Spec\left(\tilde{R}\right) \to Y$ over $Spec(R)$ with $j(\tilde{o}) = \{e\}$ (if $R$ is not already complete, then first extend $R$ to a complete DVR using [Gr, 0.6.8.2,3] so that $\tilde{R}$ is again a DVR). As $k$ is assumed algebraically closed, and $\tilde{R} \to R$ is unramified, then the extension of residue fields is finite and separable at the closed point of $R$, and hence an isomorphism of residue fields. Let $\hat{p} : \mathscr{E} \to Spec\left(\tilde{R}\right)$ be the pull-back of the universal family over $Y$. As the closed fibre $\mathscr{E}_0$ is a projective line, we may choose the morphism $j$ in such a way that the generic fibre $\mathscr{E}_1$ of $\hat{p}$ is also a projective line. Moreover, $\mathscr{E}$ can be considered as a smooth closed algebraic subspace of codimension 1 in $\hat{X} = X \otimes \tilde{R}$. By lemma 9, $-1 = \mathscr{E}_0^2 = \mathscr{E}_1^2$. Hence $\mathscr{E}_1$ is also an exceptional curve of the first kind. Hence by [A2, Cor 6.11] there exists a contraction morphism $\pi : \hat{X} \to \hat{X}$ over $Spec\left(\tilde{R}\right)$ which contracts $\mathscr{E}$ to a section of $\tilde{\varphi} : \tilde{X} \to Spec(R)$ where $\tilde{\varphi}$ is proper, smooth, separated and of finite type over $Spec\left(\tilde{R}\right)$. $\square$

**Lemma 9.** [KU, 9.3] *In the situation of the above Lemma, if $D, D'$ are divisors on $X$, then*

$$K_{X_k}^2 = K_{X_K}^2$$
$$(K_{X_k} \cdot D_k)_{X_k} = (K_{X_K} \cdot D_K)_{X_K}$$
$$(D_k \cdot D'_k)_{X_k} = (D_K \cdot D'_K)_{X_K}.$$

**Definition 10.** [Ko, 1.7] Let $X$ be a scheme. Let $p \in X$ be a regular point with ideal sheaf $\mathfrak{m}_p$ and residue field $k(p)$. Then $x_1, ..., x_n \in \mathfrak{m}_p$ are called local coordinates if their residue classes $\overline{x_1}, ..., \overline{x_n}$ form a $k(p)$-basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Let $D = \sum a_i D_i$ be a Weil divisor on $X$. We say that $(X, D)$ has simple normal crossings or snc at a point $p \in X$ if $X$ is regular at $p$ and there is an open neighborhood $p \in X_p \subset X$ with local coordinates $x_1, ..., x_n \in \mathfrak{m}_p$ such that $X_p \cap Supp\ D \subset (x_1 \cdot \cdots \cdot x_n = 0)$. We say that $(X, D)$ is snc if it is snc at every point.

We say that $(X, D)$ has normal crossing or nc at a point $p \in X$ if $\left(\hat{X}_K, D|_{\hat{X}_K}\right)$ is snc at $p$ where $\hat{X}_K$ denotes the completion at $p$ and $K$ is a separable closure of $k(p)$. We say that $(X, D)$ is nc if it is nc at every point.

If $(X, D)$ is defined over a perfect field, this concept is also called log smooth.

The following statement of resolution of singularities will be used.

**Theorem 11.** [CP2014, 1.1] *Let $X$ be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three.*

*There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*

*(i) $X'$ is everywhere regular*

*(ii) $\pi$ induces an isomorphism $\pi^{-1}(Reg\ X) \approx Reg\ X$*

*(iii) $\pi^{-1}(Sing\ X)$ is a strict normal crossings divisor on $X'$.*

**Definition 12.** A log resolution of a pair $(X, \Delta)$ is a proper birational morphism $f : Y \to X$ from a regular variety such that the exceptional locus $Ex(f)$ is a divisor and $f^{-1}(\Delta) \cup Exc(f)$ has simple normal crossings support.

**Definition 13.** Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair $(X, \Delta)$ has terminal (respectively Kawamata log-terminal i.e. "klt", respectively log-canonical) singularities if for any log resolution $f : Y \to X$ of $(X, \Delta)$ such that $E_i$ are exceptional curves on $Y$, then

$$K_Y + \Delta_Y = f^*(K_X + \Delta) + \sum a_i E_i$$

where $a_j > 0$ (respectively $a_j > -1$, respectively $a_j \geq -1$) and $\Delta_Y$ is the strict transform of $\Delta$.

The following ingredients of the minimal model program for surfaces in positive characteristic are used [KK94], [Tan].

**Definition 14.** Let $f : X \to Z$ be a projective birational morphism of algebraic spaces such that $f_*\mathcal{O}_X = \mathcal{O}_Z$ and $dim\ NE(X/Z) = 1$ and

$f$ contracts some divisor. Then $f$ is called a divisorial contraction. If instead $f$ contracts some subvariety of codimension $\geq 2$ and no divisors, then $f$ is called a small contraction. If $f$ is a small (resp. divisorial) contraction and $-D$ is $\mathbb{R}$-Cartier and relatively ample, then $f$ is a $D$-flipping contraction (resp. $D$-divisorial contraction).

**Lemma 15.** [KK94, 2.3.5] *Let $(S, B)$ be a log-canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C.(K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

**Theorem 16.** [Tan, 4.4] *Let $X$ be a projective normal surface and let $\Delta$ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let $H$ be an $\mathbb{R}$-cartier ample $\mathbb{R}$-divisor. Then the following assertions hold:*

*(1) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$*

*(2) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+H \geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i]$*

*(3) Each $C_i$ in (1) and (2) is rational or $C_i = B_j$ for some $B_j$ with $B_j^2 < 0$.*

**Theorem 17.** [Tan, 5.3] *Let $X$ be a projective normal surface and let $C$ be a curve in $X$ such that $r(K_X + C)$ is Cartier for some positive integer $r$.*

*(1) If $C \cdot (K_X + C) < 0$, then $C \approx \mathbb{P}^1$.*

*(2) If $C \cdot (K_X + C) = 0$, then $C \approx \mathbb{P}^1$ or $\mathcal{O}_C\left((K_X + C)^{[r]}\right) \approx \mathcal{O}_C$.*

**Definition 18.** [HK, 5.E] If $K_X + \Delta$ is an effective KLT pair in dimension 2, then for any ample divisor $H'$, we can find $h \in \mathbb{R}_{>0}$ and an $\mathbb{R}$-divisor $H \sim_{\mathbb{R}} hH'$ such that $(X, \Delta + H)$ is klt and $K_X + \Delta + H$ is nef and big. Let

$$\lambda = inf\left\{t \geq 0 \mid K_X + \Delta + tH \ is \ nef\right\}.$$

If $\lambda = 0$, then $K_X + \Delta$ is nef, and thus $(X, \Delta)$ is minimal. If $\lambda > 0$, then, by Theorem 16, there exists a $(K_X + \Delta)$-negative extremal ray $R$ such that $(K_X + \Delta + \lambda H).R = 0$. Then, c.f. [KK94, 2.3], if the corresponding contraction $\phi : X \to X'$ does not result in a log Del Pezzo surface or a birationally ruled surface, then setting $H' = \phi_* H$ and $\Delta' = \phi_* \Delta$, the divisor $K_{X'} + \Delta' + \delta H'$ is nef. Then the process may be repeated. The process either terminates at some step in a log minimal model or one of the aforementioned surfaces since there are no flips in dimension 2. The end result is a finite sequence of real numbers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ such that $K_{X_n} + \Delta_n + \lambda_n H_n$ is nef and $X \to X_n$ is a minimal model for $(X, \Delta + \lambda_n H)$.

The proofs in this article will also make use of several vanishing theorems.

**Theorem 19.** *(Kawamata-Viehweg Vanishing* [Tan2, 2.11]*) Let* $(X, \Delta)$ *be a projective klt surface (over an algebraically closed field of characteristic* $p > 0$*) where* $\Delta$ *is an effective* $\mathbb{R}$*-divisor. Let* $N$ *be a nef* $\mathbb{R}$*-cartier* $\mathbb{R}$*-divisor with* $\nu(X, N) \geq 1$*. Let* $D$ *be a* $\mathbb{Q}$*-Cartier* $\mathbb{Z}$*-divisor such that* $D - (K_X - \Delta)$ *is nef and big. Then there exists a positive real number* $r(\Delta, D, N)$ *such that* $H^i(X, \mathcal{O}_X(D + rN + N')) = 0$ *for* *every $i > 0$, every positive real number $r \geq r(\Delta, D, N)$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $N'$ such that $rN + N'$ is a Cartier divisor.*

**Theorem 20.** [Ek, II.6] *Let $X$ be a minimal surface of general type and let $\mathcal{L}$ be an invertible sheaf that is numerically equivalent to $\omega_S^{\otimes i}$ for some $i \geq 1$. Then $H^1(X, \mathcal{L}^{\otimes 2}) = 0$ except possibly for certain surfaces in characteristic 2 with $\chi(\mathcal{O}_X) \leq 1$ .*

**Theorem 21.** [Ter] *Let $X$ be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let $D$ be a big and nef Cartier divisor on $X$. Assume that either*

*(1) $\kappa(X) \neq 2$ and $X$ is not quasi-elliptic with $\kappa(X) = 1$; or*

*(2) $X$ is of general type with*

*$p \geq 3$ and $(D^2) > vol(X)$ or*

*$p = 2$ and $(D^2) > max\{vol(X), vol(X) - 3\chi(\mathcal{O}_X) + 2\}$.*

*Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

*for all $i > 0$.*

The following properties of base change will be used:

**Theorem 22.** [Har, III.10.2] *Let $f : X \to Y$ be a morphism of schemes of finite type over a field $k$. Then $f$ is smooth of relative dimension $n$ if and only if :*

*(1) $f$ is flat; and*

*(2) for each point $y \in Y$, let $X_{\overline{y}} = X_y \otimes_{k(y)} k(y)^-$ where $k(y)^-$ is the algebraic closure of $k(y)$. Then $X_{\overline{y}}$ is equidimensional of dimension $n$ and regular. (We say the "fibres of $f$ are geometrically regular of equidimension $n$.")*

**Theorem 23.** [GW, 6.28] *Let $k$ be a field, $X$ a $k$-scheme locally of finite type, and let $x \in X$ be a closed point. Let $d \geq 0$ be an integer. We fix an algebraically closed extension $K$ of $k$, and write $X_K = X \otimes_k K$. The following are equivalent:*

*(i) The $k$-scheme $X$ is smooth of relative dimension $d$ at $x$.*

*(ii) For every point $\overline{x} \in X_K$ lying over $x$, $X_K$ is smooth of relative dimension $d$ at $\overline{x}$.*

*(iii) For every point $\overline{x} \in X_K$ lying over $x$, the completed local ring $\hat{\mathcal{O}}_{X_K,\overline{x}}$ is isomorphic to a ring of formal power series $K[[T_1, ..., T_d]]$ over $K$.*

*(iv) For every point $\overline{x} \in X_K$ lying over $x$, the local ring $\mathcal{O}_{X_K,\overline{x}}$ is regular and has dimension $d$.*

*(v) The equalities $dim_{\kappa(x)} T_x(X/k) = dim \mathcal{O}_{X,x} = d$ hold.*

*If these conditions are satisfied, then*

*(vi) The local ring $\mathcal{O}_{X,x}$ is regular and has dimension $d$.*

*Furthermore, if $\kappa(x) = k$, then the final condition implies the other ones.*

As a consequence of the above theorems,

**Proposition 24.** *Let $(X, \Delta)$ be a terminal (resp. klt) pair which is simple normal crossings over a DVR $R$ with perfect residue field $k$ and perfect fraction field $K$. Let $(X', \Delta')$ denote the pair on the base change to a complete DVR $R'$ with algebraically closed residue field. Then $(X'_k, \Delta'_k)$ and $(X'_K, \Delta'_K)$ are log-smooth and terminal (resp. klt.)*

*Proof.* By adjunction $(X_k, \Delta_k)$ and $(X_K, \Delta_K)$ are terminal (resp. klt) and snc by definition 10. Since smoothness is preserved by base-change, then all strata of $(X', \Delta')$ are smooth. Also, $(X'_k, \Delta'_k)$ is by definition log smooth after base change to the algebraic closure of $k$, since the algebraic closure of a perfect field is equal to its separable closure. Thus $(X', \Delta')$ is log-smooth.

Next, since $k$ is assumed perfect, the notions of smooth and regular coincide. Thus by Theorems 23, and 22 we have that the base change $X \to X'$ over the algebraic closure $k'$ is smooth of relative dimension 0, hence an étale base change. Thus by [Ko, 2.14,2.15], $(X_k, \Delta_k)$ and $(X_K, \Delta_K)$ are terminal (resp. klt) iff $(X'_k, \Delta'_k)$ and $(X'_K, \Delta'_K)$ are terminal (resp. klt). $\square$

**Definition 25.** [Laz] Let $D$ be a pseudoeffective $\mathbb{R}$-divisor on a normal projective variety $X$. The diminished base locus is defined as

$$\mathbb{B}_{-}(D) := \bigcup_{\substack{A \ \ Ample \\ D+A \ \ \mathbb{Q}-Cartter}} \mathbb{B}(D+A)$$

where $\mathbb{B}(D+A) = \bigcap_{n \geq 1} Bs(n(D+A))$ is the stable base locus.

**Definition 26.** Let $D$ be a big Cartier divisor. Let $F_m$ be the fixed divisor $|mD|_{fix}$. Then $F_{m+n} \leq F_m + F_n$ and the limit

$$N_\sigma(D) := \lim_{m\to\infty} \frac{1}{m} F_m$$

exists as an $\mathbb{R}$-divisor. We have $mN_\sigma(D) \leq F_m$ and for the $\mathbb{R}$-divisor $P_\sigma(D) := D - N_\sigma(D)$, we have an isomorphism

$$H^0(X, \mathcal{O}_X(mD)) \approx H^0(X, \mathcal{O}_X(\llcorner mP_\sigma(D) \lrcorner))$$

for any $m > 0$. The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the sectional decomposition.

*Remark* 27. If $D$ is not big, then $|mD|$ may be empty for certain positive integers $m$, and thus, in defining $N_\sigma(D)$, it is necessary to consider only the semigroup $\mathbb{N}(D)$ of $m$ such that $|mD| \neq \emptyset$ for such $D$. In this article, only the sectional decomposition of big divisors is considered.

The following results on arithmetic schemes are also used:

**Theorem 28.** *Let $S$ be an affine Dedekind scheme and $f : X \to S$ a projective morphism. Let $\mathcal{L}$ be an invertible sheaf on $X$ such that $\mathcal{L}_s$ is nef for every closed point $s \in S$. Then $\mathcal{L}$ is nef.*

*Proof.* By [Liu, 5.3.24], the statement holds when nef is replaced by ample. Now, c.f. [Laz, 1.4.10], it suffices to choose an ample divisor $H$ which restricts to $X_s$, so that $D_s + \epsilon H_s$ is ample for all sufficiently small $\epsilon$. Then $D + \epsilon H$ is ample for all sufficiently small $\epsilon$, and so $D$ is nef. ∎

**Theorem 29.** *([Liu, 5.3.20, 5.3.22]) Let $S = Spec\ \mathcal{O}_K$ be the spectrum of a discrete valuation ring $\mathcal{O}_K$, with generic point $\eta$ and closed point $s$. Let $f : X \to S$ be a projective morphism and $\mathcal{F}$ a coherent sheaf on $X$ that is flat over $S$. Fix $p \geq 0$. TFAE.*

*(1) We have equality $dim_{k(s)} H^p(X_s, \mathcal{F}_s) = dim_{k(\eta)} H^p(X_\eta, \mathcal{F}_\eta)$.*

*(2) $H^p(X, \mathcal{F})$ is free over $\mathcal{O}_K$ and the canonical homomorphism $H^p(X, \mathcal{F}) \otimes_{\mathcal{O}_K} k(s) \to H^p(X_s, \mathcal{F}_s)$ is a bijection.*

*(2) The $\mathcal{O}_K-$module $H^{p+1}(X, \mathcal{F})$ is torsion-free.*

*Also, regardless of whether the above hold, $\chi_{k(s)}(\mathcal{F}_s) = \chi_{k(\eta)}(\mathcal{F}_\eta)$.*

**Theorem 30.** *([Har, III.12.11] Let $f : X \to Y$ be a projective morphism of Noetherian Schemes, and let $F$ be a coherent sheaf on $X$, flat over $Y$.Let $y$ be a point of $Y$. Then*

*(a) if the natural map $\varphi^i(y) : R^i f_*(F) \otimes k(y) \to H^i(X_y, F_y)$ is surjective, then it is an isomorphism, and the same is true for all $y'$ in a suitable neighborhood of $y$;*

*(b) Assume $\varphi^i(y)$ is surjective. Then TFAE: (i) $\varphi^{i-1}(y)$ is also surjective. (ii) $R^i f_*(F)$ is locally free in a neighborhood of $y$.*

**Theorem 31.** [Mum, Sec 5] *Let $f : X \to Y$ be a proper morphism of noetherian schemes with $Y = Spec\ A$ affine, and $\mathcal{F}$ a coherent sheaf on $X$, flat over $Y$. There is a finite complex $K^\bullet : 0 \to K^0 \to ... \to K^n \to 0$ of finitely generated projective (hence free if $A$ is local) $A$-modules and an isomorphism of functors*

$H^p(X \times_Y Spec\ B, \mathcal{F} \otimes_A B) \approx H^p(K^\bullet \otimes_A B)$, $(p \geq 0)$ *on the category of A-algebras B.*

**Corollary 32.** *([Oss, 2.2]) Let* $f : X \to S$ *be a proper morphism of Noetherian schemes, and* $F$ *a coherent* $\mathcal{O}_X$*module, flat over* $S = Spec\ R$ *for a DVR* $R$*,. Then any jumps in the function*

$$s \mapsto dim_{k(s)} H^i(X_s, F_s)$$

*are necessarily paired in consecutive indices* $i$ *and* $i+1$*. Given any point* $s \in S$*, we can thus inductively determine for which pairs* $i, i+1$ *such a jump occurs, and by how much.*

*Moreover, in the case that* $S = Spec\ R$ *for* $R$ *a DVR (or more generally regular of dimension 1), we find that* $R^{i+1} f_* F$ *has torsion if and only if such a jump occurs for* $i, i+1$*.*

*Proof.* (I've included the referenced proof here, since it seems to be a collection of personal notes, and I can't find a published reference). The key point is the following: applying the above Theorem 31 we find a "free" complex $(K^\bullet, d^p : K^p \to K^{p+1})$ as in Theorem 31.

Then, following Mumford, [Mum, Lemma on pg50] $dim_{k(y)} H^p(X_y, \mathcal{F}_y) = dim_{k(y)}[ker(d^p \otimes_A k(y))]$

$$-dim_{k(y)}\left[Im\left(d^{p+1} \otimes_A k(y)\right)\right]$$

$$= dim_{k(y)}[K^p \otimes k(y)] - dim_{k(y)}[Im(d^p \otimes k(y))]$$

$$-dim_{k(y)}\left[Im\left(d^{p-1} \otimes k(y)\right)\right].$$

By freeness, the first term is constant on $Y$. As in the cited lemma, for any $p \geq 0$, the function $dim_{k(y)}\left[Im\left(d^p \otimes k\left(y\right)\right)\right]$ is lower semi-continuous on $Y$. Thus if $dim_{k(y)}\left[Im\left(d^{p-1} \otimes k\left(y\right)\right)\right]$ jumps down, it must affect both $dim_{k(y)}H^{p-1}\left(X_y, F_y\right)$ and $dim_{k(y)}H^p\left(X_y, F_y\right)$ (since by lower-semicontinuity, it is impossible to counter with a drop of rank at the generic point on $H^p$). This gives the first part of the theorem.

Now, consider the exact sequence

$$0 \to im\ d^p \to K^{p+1} \to K^{p+1}/\left(im\ d^p\right) \to 0.$$

C.f. [Liu, exc 1.2.15], $K^{p+1}/\left(im\ d^p\right)$ torsion-free, implies $im\ d^p$ is flat. A finite, flat $R$-module over reduced noetherian ring is flat iff the rank $M \otimes_R k\left(\mathfrak{p}\right)$ is locally constant for $\mathfrak{p} \in Spec\ A$ (c.f. for example [Gra, I.7.15]). Under the exact sequence:

$$0 \to ker\left(d^{p+1}\right)/im\left(d^p\right) \to K^{p+1}/\left(im\ d^p\right) \to K^{p+1}/\left(ker\ d^{p+1}\right) \to 0$$

the last term injects into $K^{p+2}$, which is free, and therefore the last term has no torsion. As $ker\left(d^{p+1}\right)/im\left(d^p\right) \approx R^{p+1}f_*F$, by Theorem 31, if that term is torsion free, then $K^{p+1}/\left(im\ d^p\right)$ is torsion free, and thus $im\ d^p$ is flat, which implies that $dim_{k(y)}H^p\left(X_y, \mathcal{F}_y\right)$ is locally constant by the above. □

## 3. Effective KLT Kawamata Viehweg Vanishing

In this modification I note that, a simple modification of Tanaka's theorem 3 gives you version of the (log) Kawamata Viehweg Vanishing in which the multiple of the nef divisor is at least computable. Note that there is a similar recent result of [DF], which is more restrictive in that it requires a smooth variety rather than a normal projective klt pair.

**Theorem 33.** *( Effective KLT Kawamata-Viehweg Vanishing for normal surfaces) Let $(X,\Delta)$ be a normal projective klt surface (over an algebraically closed field of characteristic $p>2$) where $\Delta$ is an effective $\mathbb{R}$-divisor. Let $N$ be a nef and big $\mathbb{R}$-cartier $\mathbb{R}$-divisor with $N^2 > vol(X)$. Let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D-(K_X-\Delta)$ is nef and big. Then $H^i(X, \mathcal{O}_X(D+rN+N'))=0$ for every $i>0$, every positive real number $r\geq 1$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $N'$ such that $rN+N'$ is a Cartier divisor.*

*Proof.* Verbatim from [Tan2, 2.11] except that Weak Kodaira Vanishing [Tan2, 2.4] in the proof of [Tan2, 2.6] is replaced by Terakawa's theorem 21.

□

## 4. Intersection Theory on $Y/R$.

In this section, $Y/R$ is a proper algebraic space of relative dimension $d=2$ with geometrically connected fibers where $R$ is a discrete valuation ring with residue field $k$ of characteristic $p>0$ and fraction field $K$ of characteristic 0. (See for e.g. The compact Shimura Varieties mentioned from [Suh, 0.1].

If $Y/R$ is smooth, then lemma 9 applies to show that for any divisors $C, D$ extending to both fibers, we can just define $C.D = (C|_{X_k}.D|_{X_k})_{X_k}$. On the other hand, if $Y/R$ is merely normal and proper, but is actually a scheme, the resolution of singularities, theorem 11, holds. Thus the intersection theory can be defined as in [Tan, Def 3.1]: $f : X' \to X$ is a resolution, and $C.D = f^*C.f^*D$ for two divisors $C, D$ on $Y/R$. By properness, this intersection extends by linearity to Weil divisors with $\mathbb{Q}$ or $\mathbb{R}$ coefficients. Numerical equivalence and $N^1(X)_{\mathbb{Q},\mathbb{R}}$ are then defined as usual.

Note the two different hypothesis here: The invariance of plurigenera theorem requires a proper, smooth algebraic space of relative dimension 2, while the existence of minimal models requires the intermediate steps of the minimal model program to have some intersection theory, and thus be schemes (but not necessarily smooth).

## 5. Proof of Main Result

The proof of the main result goes in several steps. The first step is to show that under more restrictive assumptions on the pair $(X, \Delta)$, that the invariance of plurigenera holds. The following is similar to [Tan3, 7.3] but without an ample boundary. The proof is similar to the Theorem [Suh, 1.2.1(ii)], where empty boundary is considered and there is, in addition, assumed a $W_2$ lifting hypothesis.

**Proposition 34.** *Let $(X, \Delta)$ be a terminal log smooth pair of relative dimension 2 over a DVR $R$ with algebraically closed residue field $k$ of characteristic $p > 2$ and perfect fraction field $K$. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\nu(K_X + \Delta) \neq 1$. Assume that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$. Then there exists an $m_0$ such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0(m(K_{X_k} + \Delta_k)) = h^0(m(K_{X_K} + \Delta_K)).$$

*If $K_X + \Delta$ is big, then $m_0$ is computable. If serre duality holds on $X$, then the same holds for $\nu(K_X + \Delta) = 1$.*

The proof of proposition 34, is given in the following claims.

*Claim.* Assumptions as above, after passing to an extension $R'$ of $R$, there is a proper, smooth algebraic space $X^{min}/R'$ and an $R'$ morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of $(-1)$ curves and $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef.

*Proof.* As $k$ is algebraically closed of characteristic $p > 0$, then by the Cone Theorem 16,

$$\overline{NE}(X_k) = \overline{NE}(X_k)_{K_{X_k}+\Delta_k \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

with each $C_i$ is rational or $C_i = B_j$ for some $B_j$ a component of $\Delta$ with $B_j^2 < 0$. Under the assumption that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$, then actually each $C_i$ is rational and is not a component of $\Delta_k$. Thus $C_i.\Delta_k \geq 0$ so $C_i.K_{X_k} < 0$, and since $N_\sigma(K_{X_k} + \Delta_k)$ is a $\mathbb{Q}$-divisor by [Tan, 7.1], then Theorem 17 implies $C_i \approx \mathbb{P}^1$. By Lemma 15, $C_i$ can be contracted to a log-canonical point, which is actually a smooth point under the terminal assumption. Thus $C_i$ is an exceptional curve of the first kind, so it is possible to apply Lemma 8.

By Lemma 8, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \to \tilde{X}$ induces the contraction of $C_i$ on $X_k$ and $X_K$, and further $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since $k$ is algebraically closed, it induces identity on residue fields. Now I need to work with $\tilde{X}$. In order to apply the Cone Theorem 16 again, I need $\tilde{X}_k$ to be projective, but a smooth algebraic space of dimension 2, proper, separated, and of finite type over an algebraically closed field is projective, so $\tilde{X}_k$ is projective. Thus the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no problem in extending $R$. As the Picard number drops each step, there are only finitely many steps. □

*Claim* 35. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef. (In the general type case, there is another argument, c.f. two claims down.)

*Proof.* It suffices to apply 28 to $X_k^{min}$, and then restrict to $X_K$. □

Now we proceed by cases depending on the Kodaira dimension.

*Claim* 36. (Case 1: $\nu = 2$) In the case that the special fiber is big, there is a computable $m_0$ such that for $m_0|m$, we have

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

*Proof.* By the above, we achieve $X^{min}$ such that $K_X + \Delta$ is nef on both $X_k^{min}$ and $X_K^{min}$. By the Kawamata Viehweg Vanishing Theorem 33 applied to the special fiber (and the semicontinuity theorem) there exists a computable $m_0 \gg 0$ such that for $m_0 < m$ and $i > 0$, we have

$$h^i\left(m\left(K_{X_K} + \Delta_K\right)\right) = h^i\left(m\left(K_{X_k} + \Delta_k\right)\right) = 0.$$

In fact, if $\Delta = 0$, then Ekedahl's vanishing Theorem 20 can be applied. Thus, applying the invariance of Euler characteristic (Theorem 29), and birational invariance of the plurigenera, it follows that for $m > m_0$,

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Delta_K\right)\right).$$

□

*Claim* 37. If Serre duality holds on $X/R$, then for sufficiently large $m$ and any kodaira dimension, we have

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Delta_K\right)\right).$$

*Proof.* By the previous claim, we achieve an $\pi' : X^{min} \to R'$ and an $R'$ morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of $(-1)$ curves and both $K_{X_k^{min}} + \Delta_{X_k^{min}}$ and $K_{X_K^{min}} + \Delta_{X_K^{min}}$ are nef. Now, by the Theorems 29 and 32 it suffices to show that $H^1\left(X, \mathcal{O}_X\left(m\left(K_X+\Delta\right)\right)\right)$ is torsion free or that $R^1\pi'_*\mathcal{O}_X\left(m\left(K_X+\Delta\right)\right)$ is torsion free. It seems that any torsion in the second sheaf occurs at the closed point $(k)$ of $S = Spec\ R$, so it should? (this would be easier if Serre Duality held over R!) suffice to show that the duals of these $\mathcal{O}_K-$modules are $(k)-$divisible.

C.f. Suh we may base to assume $R'$ is unramified, and thus c.f. [Suh, 1.2.1], $X_k^{min}$ lifts to $W_2$. C.f. [Tan], $K_X + \Delta$ is semiample, so $|m\left(K_{X_k}+\Delta_k\right)|$ contains a smooth curve for $m \gg 0$ which therefore lifts to $W_2(k)$. Let $L := m\left(K_X+\Delta\right)$. As we are in the terminal category, and $L$ has no stable base locus, then point blowups preserve our $W_2(k)$ lifting of $X_k$, and also global generation is preserved, c.f. [Suh] or Lietdke. Thus the reductions of [EV, 5.12.a from the online lecture notes version] hold in this setting, and this gives a surjectivity $H^1\left(L_k^* \otimes N^*_{X_k/X}\right) \to H^1\left(L_k^*\right)$. Now for $m \gg 0$, $H^2(L) = H^2\left(L \otimes N_{X_k/X}\right) = 0$, so by Theorem 29, $H^1(L^*) \otimes k \approx H^1(X_k, L_k^*)$, $H^1(L^*(-X_k)) \otimes k \approx H^1\left(X_k, L_k \otimes N^*_{X_k/X}\right)$. Applying Nakayama's Lemma to the diagram

$$\begin{array}{ccc} H^1\left(L^*\left(-X_k\right)\right) & & H^1\left(L^*\right) \\ \downarrow & & \downarrow \\ H^1\left(X_k, L_k \otimes N^*_{X_k/X}\right) & \longrightarrow & H^1\left(X_k, L_k^*\right) \end{array}$$

gives a surjection $H^1(L^*(-X_k)) \twoheadrightarrow H^1(L^*)$ which concludes the proof.

□

*Claim* 38. (Case 2: $\nu = 0$) In the case that the special fiber has $\nu(K_{X_k} + \Delta_k) = 0$, there is a computable $m_0$ such that for $m_0 | m$, we have

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

*Proof.* As before, we achieve an $\pi' : X^{min} \to R'$ and an $R'$ morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of $(-1)$ curves and both $K_{X_k^{min}} + \Delta_{X_k^{min}}$ and $K_{X_K^{min}} + \Delta_{X_K^{min}}$ are nef.

Applying [Tan], we find that $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is semi-ample. Under the pseudo-effective assumption, and since $\nu\left(K_{X_k^{min}} + \Delta_{X_k^{min}}\right) = 0$, but $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef, then actually $K_{X_k^{min}} + \Delta_{X_k^{min}} \equiv 0$. A high multiple of a semi-ample, and numerically trivial bundle is globally generated and numerically trivial, hence $\mathcal{O}_{X_k^{min}}\left(m\left(K_{X_k^{min}}+\Delta_{X_k^{min}}\right)\right)=\mathcal{O}_{X_k^{min}}$, $m_k \gg 0$. Actually the same holds on both fibers, since the abundance has been proven over a field $k$. Thus, perhaps taking $m' = m_k \cdot m_K$, then the conclusion follows from Theorem 32 and cohomological flatness of the smooth map $X^{min} \to R'$, c.f. [Liu, Exc. 5.3.14]. ☐

The next step is to remove the restriction on the base locus from proposition 34.

**Corollary 39.** *Let $(X, \Delta)$ be a klt pair of relative dimension 2 over a DVR $R$ with perfect residue field $k$ of characteristic $p > 0$ and perfect fraction field $K$. Assume that $K_X + \Delta$ is, big, $\mathbb{Q}$-Cartier, and simple normal crossings over $R$. Then, there exists a (computable) $m_0$ depending on the intersection numbers, such that for $m \in m_0\mathbb{Z}^+$ ,*

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Delta_K\right)\right)$$

*thus, after possibly extending $R$, the canonical ring $R(K_X + \Delta)$ is finitely generated over $R$. If $K_X + \Delta$ is pseudo-effective(not necessarily big), a contraction of a $K_{X_k} + \Delta_k$-negative extremal ray cut out by an ample $H$ on $R$ induces a contraction of a $K_X + \Delta$ negative extremal ray. If the contraction is a flipping contraction, then the flip exists.*

*Proof.* Since the hypothesis and conclusion are preserved by base change, then extending $R$, we may assume that $k$ is algebraically closed and that $R$ is complete. The proof follows similarly to [HMX2010, 1.6] except that Proposition 34 is used in place of Kawamata Viehweg Vanishing.

Recall that the log-canonical ring of $K_{X_k}+\Delta_k$ is finitely generated (c.f. [Tan, 7.1]) so $N_\sigma\left(K_{X_k}+\Delta_k\right)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k=\Delta_k-\Delta_k\wedge N_\sigma\left(K_{X_k}+\Delta_k\right)$$

and let $0\leq\Theta\leq\Delta$ be a $\mathbb{Q}$-divisor on $X/R$ such that, by log-smoothness $\Theta|_{X_k}=\Theta_k$. Replacing $(X,\Delta)$ by a blow-up, assume $(X,\Delta)$ is terminal. By definition of $N_\sigma$, letting $m\gg0$ to fit the hypothesis of Proposition 34, and sufficiently divisible such that $m\left(K_{X_k}+\Theta_k\right)$ is integral, then

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_k}+\Theta_k\right)\right)$$

and furthermore, by Proposition 34,

$$h^0\left(m\left(K_{X_k}+\Theta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Theta_K\right)\right).$$

As $\Theta\leq\Delta$, the first part of the theorem follows by semicontinuity.

Now by the theory of surfaces, there is $m\gg0$, such that $\bigoplus_{m\geq0}H^0\left(ml\left(K_{X_k}+\Delta_k\right)\right)$ is generated in degree 1, so that there is a surjection

$$S^lH^0\left(m\left(K_{X_k}+\Delta_k\right)\right)\twoheadrightarrow H^0\left(ml\left(K_{X_k}+\Delta_k\right)\right).$$

Consider the following diagram:

$$\begin{array}{ccc}
S^lH^0\left(m\left(K_X+\Delta\right)/R\right) & \xrightarrow{\alpha} & H^0\left(ml\left(K_X+\Delta\right)/R\right)\\
\downarrow & & \downarrow\\
S^lH^0\left(m\left(K_{X_k}+\Delta_k\right)\right) & \xrightarrow{\twoheadrightarrow} & H^0\left(ml\left(K_{X_k}+\Delta_k\right)\right)
\end{array}$$

The vertical maps are surjective by the first part of the theorem (and applying Nakayama's lemma to pull back the generators of the $k$-modules). Now again using, Nakayama's Lemma [GH, Chapter 5.3], surjectivity of the bottom map implies surjectivity of $\alpha$, and thus $R(K_X + \Delta)$ is finitely generated over $R$.

Finally suppose that there is a contraction of a curve on the closed fiber $(X_k, \Delta_k)$ corresponding to a $K_{X_k} + \Delta_k$ -negative extremal ray $\Sigma_k$. If necessary, extend $R$ to be complete and $k$ algebraically closed. Let $H_k = H|_{X_k}$ be such that $(K_{X_k} + \Delta_k + H_k).\Sigma_k = 0$. The goal is to show that

$$R(K_X + \Delta + H/R)$$

is finitely generated so that there exists a contraction defined by $X_1 = Proj\ R(K_X + \Delta + H/R)$. However, as $H$ is ample, then under the assumption that $K_{X_k} + \Delta_k$ is pseudo-effective, $K_{X_k} + \Delta_k + H_k$ is actually big, and so this follows.

□

**Corollary 40.** *Let $(X, \Delta)$ be a klt pair of relative dimension 2 over a DVR $R$ with perfect residue field $k$ of characteristic $p > 0$ and perfect fraction field $K$. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and simple normal crossings over $R$. If $A_1, ..., A_m$ are any ample divisors, then the ring $R(X, K_X + \Delta + A_1, ..., K_X + \Delta + A_n)$ is finitely generated over $R$.*

*Proof.* C.f. the proof of [CZ, 2.11], it suffices to prove that if $G = \sum_{i=1}^{k} m_i (a (K_X + \Delta_i + A_i))$, fixed $a \in \mathbb{N}$, and $m_1, ..., m_k \geq 0$ and $l \in$

$\{1, ..., l\}$ then there exists $m_l$ such that the following maps are surjective

$$H^0\left(X, \mathcal{O}_X\left(G - a\left(K_X + \Delta - A_l\right)\right)\right) \otimes H^0\left(X, \mathcal{O}_X\left(a\left(K_X + \Delta + A_l\right)\right)\right) \to H^0\left(X, \mathcal{O}_X\left(G\right)\right)$$

is surjective. By Nakayama's lemma applied as above, it suffices to show the surjectivity on the central fiber.

Applying [Tan], there exists a minimal model $(X'_k, \Delta'_k)$ for $(X_k, \Delta_k)$, such that $K_{X_k} + \Delta_k$ is nef and semi-ample. Thus there is $m \gg 0$ such that $m\left(K_{X_k} + \Delta_k\right)$ is both bpf and $m - 2$ is large enough so the effective Kawamata Viehweg vanishing Theorem 33applies. Now the proof of [CZ, 2.7] applies. □

Thus:

**Corollary 41.** [CL, 3.8]*Let $(X, \Delta)$ be a klt pair of relative dimension 2 over a DVR $R$ with perfect residue field $k$ of characteristic $p > 0$ and perfect fraction field $K$. Assume that $\Delta$ is big. If $K_X + \Delta$ is nef, then it is semi-ample.*

**Definition 42.** [CL, def 4.1] Let $W$ a finite dimensional real vector space, $C \subset W$ a closed convex cone spanning $W$, and $v \in W$. The visible boundary of $C$ from $v$ is

$$V = \{w \in \partial C \mid [v, w] \cap C = \{w\}\}.$$

**Corollary 43.** [CL, 4.2] *(Kawamata's Rationality, Cone and Contraction Theorem) Let $(X, \Delta)$ be a klt pair of relative dimension 2, proper* *over a DVR $R$ with perfect residue field $k$ of characteristic $p > 0$ and perfect fraction field $K$. Assume either that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and simple normal crossings over $R$, or is obtained after a finite number of steps of the minimal model program from such a $\mathbb{Q}$-divisor. Let $V$ be the visible boundary of $Nef\left(X/R\right)$ from $v_0 \in N^1\left(X\right)_{\mathbb{R}}$ of the divisor $K_X + \Delta$. Then every compact subset $F \subset relint\ V$ is contained in a union of finitely many supporting rational hyperplanes, and every $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ with class in relint $V$ is semiample.*

*Proof.* As in the cited theorem, noting it only requires the finite generation of Corollary 40. □

**Lemma 44.** [CL, 6.2]*Let $(X, \Delta)$ be a klt pair of relative dimension 2 over a DVR $R$ with perfect residue field $k$ of characteristic $p > 0$ and perfect fraction field $K$. Suppose $(X, \Delta)$ is not nef, and $A$ is a big $\mathbb{Q}$-divisor such that $(X, \Delta + A)$ is klt, and $K_X + \Delta + A$ is nef. Let $\lambda$ be the nef threshold. Then $\lambda \in \mathbb{Q}^+$, and there is $R \subset \overline{NE}\left(X\right)$ with $\left(K_X + \Delta + \lambda A\right).R = 0$ and $\left(K_X + \Delta\right).R < 0$.*

*Proof.* As in the cited theorem, the only result being needed is the finite generation Corollary 40. □

**Corollary 45.** *Let $(X, \Delta)$ be a klt pair of relative dimension 2 over a DVR $R$ with perfect residue field $k$ of characteristic $p > 0$ and perfect fraction field $K$. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, pseudo-effective, and simple normal crossings over $R$. Then the minimal model of $(X, \Delta)$ exists.*

*Proof.* Let $A$ be a sufficiently general ample divisor. Consider a minimal model program with scaling of $A_1 = A$ run (slightly different from usual) as follows. Let

$$\lambda = inf\left\{t \geq 0 \mid K_X + \Delta + tA_1 \ is \ nef \ on \ X_k\right\}.$$

If $\lambda = 0$, then we are done. Otherwise, since nefness on $X_k$ implies on $X$ ([Liu, 5.3.24] and [Laz, 1.4.10]), applying Lemma 44, there exists a $(K_X + \Delta)$-negative extremal ray $C$ such that $(K_X + \Delta + \lambda A).C = 0$. Take the contraction on $(X, \Delta + \lambda A)/R$ given by the previous theorem and replace $(X_1, \Delta_1 + \lambda_1 A)$ by the corresponding flip or contraction $\phi : (X_1, \Delta_1 + \lambda_1 A) \to (X_2, \Delta_2 + \lambda_2 A)/R$. Let $A_2 = \phi_* A_1$ and $\Delta_2 = \phi_* \Delta_1$. Then $K_{X_2} + \Delta_2 + \lambda A_2$ is nef on both fibers, so noting that:

$$R_i = R\left(X_i, K_{X_i} + \Delta_i + \lambda_i A_i\right)$$

$$\approx R\left(X_1, K_{X_1} + \Delta_1 + \lambda_i A\right)$$

we repeat the process, since we can argue the contraction exists via $X_1$, which is log-smooth. Now the goal is to show the sequence of $\lambda_i$ always decreases after a finite number of indices (i.e. there is no infinite sequence of flips). This follows from the finiteness of Weak Log Canonical Models on the special fiber (essentially just a simplified version of the argument in [BCHM], noting that the two required results: existence of log terminal models and non-vanishing theorem hold on the special fiber via [Tan]) as each flipping contract induces a weak log canonical model on the special fiber. As there can be no infinite sequence of contractions, the minimal model program with scaling must therefore terminate. Note there is a more direct proof in the case of $\nu\left(K_{X_k}+\Delta_k\right)\neq 1$ via the techniques of [CL], which would also hold assuming the invariance of plurigenera for $\nu\left(K_{X_k}+\Delta_k\right)=\kappa\left(K_{X_k}+\Delta_k\right)=1$. □

The following is another Corollary of Theorem 34, for pairs not of general type. This is a result of the proof of Theorem 34 not being affected by adding an ample divisor $H$. By abundance for log surfaces, this gives a log version of [KU].

**Corollary 46.** *Let $(X,\Delta)$ be a klt log-smooth pair of relative dimension 2 over a DVR $R$ with residue field $k$ and fraction field $K$ such that $k$ is algebraically closed of characteristic $p>0$. Assume $K_X+\Delta$ is pseudo-effective. Then the numerical kodaira dimension satisfy*

$$\nu\left(K_{X_K}+\Delta_K\right)=\nu\left(K_{X_k}+\Delta_k\right).$$

*As a result of abundance for log surfaces, the log kodaira dimension also satisfy*

$$\kappa\left(K_{X_K}+\Delta_K\right)=\kappa\left(K_{X_k}+\Delta_k\right).$$

*Proof.* For any ample $A$, there exists an $m_0$ such that for $m>m_0$, $K_X+\Delta+\frac{1}{m}A=K_X+\Delta'$ is klt and and big (since $K_X+\Delta$ is assumed pseudo-effective). Thus the result follows from Corollary 39. □

## 6. Supplementary Algebra Results

6.1. **Discrete Valuation Rings.** This section contains some basic results on discrete valuation rings useful for understanding Lemma 8.

**Definition 47.** A discrete valuation ring (DVR for short) $R$ is an integral domain which is an integrally closed noetherian local ring with Krull dimension one.

**Lemma 48.** *Let $A$ be a discrete valuation ring with fraction field $K$. Let $L/K$ be any finite separable extension, and let $B$ denote the integral closure of $A$ in $L$. Then $B$ is a finite $A$-module, and if $A$ is complete, then so is $B$, and $B$ is a DVR. That is, finite extensions are complete and unsplit.*

**Definition 49.** Let $A, B$ be Noetherian local rings. A local homomorphism $A \to B$ is said to be an unramified homomorphism of local rings if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,

(2) $\kappa(\mathfrak{m}_B)$ is a finite separable extension of $\kappa(\mathfrak{m}_A)$, and

(3) $B$ is essentially of finite type over $A$ (i.e. $B$ is the localization of a finite type $A$-algebra at a prime.

**Definition 50.** Let $A, B$ be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be an étale homomorphism of local rings if it is a flat and unramified homomorphism of local rings. If $Y$ is a locally Noetherian scheme, and $f : X \to Y$ is a morphism of schemes which is locally of finite type, then $f$ is said to be étale if it is étale at all its points.

**Lemma 51.** [Deb, 6.14] *Let $A$ be a noetherian integrally closed local ring with fraction field $K$ and set $S = Spec\,(A)$. Let $\phi : X \to S$ be an étale cover. Then $X$ is also normal, and in particular, it can be written as the coproduct of its (finitely many) irreducible components. Furthermore, given a connected component $X_0$ of $X$, the induced étale cover $X_0 \to S$ is the normalization of $S$ in $k\,(S) \hookrightarrow k\,(X_0)$.*

From the above lemmas, it is clear that an étale cover of a complete DVR results is a complete DVR.

**Lemma 52.** [Bo, 2.2.14] *Let $f : X \to S$ be a smooth morphism (of schemes). Let $s$ be a point of $S$, and let $x$ be a closed point of the fibre $X_s = X \times_S Spec\; k\,(s)$ such that $k\,(x)$ is a separable extension of $k\,(s)$. Then there exists an étale morphism $g : S' \to S$ and a point $s' \in S'$ above $s$ such that the morphism $f' : X \times_S S' \to S'$ obtained from $f$ by the base change $S' \to S$ admits a section $h : S' \to X \times_S S'$, where $h\,(s')$ lies above $x$, and where $k\,(s') = k\,(x)$.*

*Remark* 53. It seems to me, that if $k$ is perfect, the proof of the above lemma goes through when $f : X \to S$ is merely smooth at $x \in X$ and $f : X \to S$ is locally of finite type.

6.2. **Riemann-Roch for Perfect Fields.** The standard Riemann-Roch theorem for algebraic surfaces is stated over an algebraically closed field. I'm sure the version for perfect fields is known if true, but since I was unable to find a reference, in this section I give necessary ingredients to extending the proof in [Har, Chapter 5.1] to perfect fields.

**Lemma 54.** [Liu, 7.3.16] *Let $X$ be a projective variety over a field $k$. Let*

$$0 \to F \to G \to H \to 0$$

*be an exact sequence of coherent sheaves on $X$. Then*

$$\chi(G) = \chi(F) + \chi(H).$$

**Theorem 55.** [Liu, 7.3.17] *Let $X$ be a projective curve over a field $k$. Let $D$ be a Cartier divisor on $X$. Then we have*

$$\chi(\mathcal{O}_X(D)) = deg\ D + \chi(\mathcal{O}_X).$$

**Definition 56.** [Liu, 4.1.2, 8.3.1]A normal Noetherian integral domain of dimension 0 or 1 is called a Dedekind domain. A normal locally Noetherian scheme of dimension 0 or 1 is called a Dedekind scheme. Let $S$ be a Dedekind Scheme. We call an integral, projective, flat $S$-scheme $\pi : X \to S$ of dimension 2 a fibered surface over $S$. If $dim\ S = 0$ then $X$ is an integral, projective, algebraic surface over a field. An irreducible Weil divisor $D$ is called horizontal if $dim\ S = 1$

and if $\pi|_D : D \to S$ is surjective. If $\pi(D)$ is reduced to a point, we say that $D$ is vertical.

**Theorem 57.** [Liu, 9.1.12]*(Intersection on a fibered surface). Let $X \to S$ be a regular fibered surface. Let $s \in S$ be a closed point. Then there exists a unique bilinear map (of $\mathbb{Z}$-modules)*

$$i_s : Div(X) \times Div_s(X) \to \mathbb{Z}$$

*which verifies the following properties:*

*(a) If $D \in Div(X)$ and $E \in Div_s(X)$ have no common component, then*

$$i_s(D, E) = \sum_x i_x(D, E)[k(x) : k(s)],$$

*where $x$ runs through the closed points of $X_s$.*

*(b) The restriction of $i_s$ to $Div_s(X) \times Div_s(X)$ is symmetric.*

*(c) $i_s(D, E) = i_s(D', E)$ if $D \sim D'$.*

*(d) If $0 < E \leq X_s$, then*

$$i_s(D, E) = deg_{k(s)}\mathcal{O}_X(D)|_E.$$

**Theorem 58.** [Liu, 9.1.37] *Let $X \to S$ be a regular fibered surface, $s \in S$ a closed point, and $E \in Div_s(X)$ such that $0 < E < X_s$ (the second inequality is an empty condition if dim $S = 0$). Then we have*

$$\omega_{E/k(s)} \approx \left(\mathcal{O}_X(E) \otimes \omega_{X/S}\right)|_E,$$

*and if $K_{X/S}$ is a canonical divisor,*

$$p_a(E) = 1 + \frac{1}{2}\left(E^2 + K_{X/S} \cdot E\right).$$

**Theorem 59.** [Har, III.7.7] *Let $X$ be a projective Cohen-Macaulay scheme of equidimension $n$ over a field $k$. Then for any locally free sheaf $\mathscr{F}$ on $X$ there are natural isomorphisms*

$$H^i(X, \mathscr{F}) \approx H^{n-i}(X, \mathscr{F}^\vee \otimes \omega_X^\circ)'$$

*where $\omega_X^\circ$ is the dualizing sheaf on $X$.*

**Theorem 60.** [BC, Cor 9.] *Let $X$ be a nonsingular projective variety of dimension $n$ over a perfect field $k$. Then $\omega_X^\circ = \Omega_X^n$.*

Finally note that the following Riemann-Roch formula holds over perfect fields. The proof is the same as the one in [Har] Chapter 5, but the field is no longer assumed algebraically closed. All the necessary results are stated above.

**Theorem 61.** *Let $D$ be a divisor on a nonsingular, projective surface $X$ over a perfect field $k$. Then*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D.(D-K) + \chi(\mathcal{O}_X).$$

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