INVARIANCE OF PLURIGENERA AND FINITE GENERATION FOR LOG SURFACES OF GENERAL TYPE IN MIXED CHARACTERISTIC

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Abstract. Log deformation invariance of plurigenera, and finite generation of the canonical ring for a general type family of relative dimension 2 over a DVR in mixed characteristic.

Contents

1. Introduction

The purpose of this article is to consider the mixed characteristic analogue of the famous theorem of Siu in the case of an algebraic variety X/R of relative dimension 2 over a DVR . Siu's theorem states

Theorem 1. [Siu] *Let* $\pi : X \to \Delta$ *be a smooth projective family of compact complex manifolds parametrized by the open unit 1-disk* Δ*.*

Assume that the family $\pi : X \to \Delta$ is of general type. Then for every positive integer m, the plurigenus $dim_{\mathbb{C}}\Gamma(X_t, mK_{X_t})$ is independent of $t \in \Delta$, where $X_t = \pi^{-1}(t)$ and K_{X_t} is the canonical line bundle of X_t.

The above theorem in characteristic 0 has been generalized to all Kodaira dimensions and to kawamata log-terminal, log smooth pairs c.f. [Siu2], [HMX2010, 1.8], and [BP].

In positive characteristic, the above theorem in general does not hold even when the fibers are surfaces with Kodaira dimension 1, since there are examples of Katsura and Ueno [KU] where a fiber with wild ramification causes the geometric genus to jump rather than being constant in the family. Similarly, Suh [Suh] has constructed counterexamples in Kodaira dimension 2 to invariance of the geometric genus.

However, in their paper, Katsura and Ueno also show that in the case of a smooth algebraic variety X/R over a DVR of relative dimension 2 with residue field k and fraction field K, then $\kappa(X_k) = \kappa(X_K)$. So the question becomes whether some asymptotic version of Siu's theorem holds in this case. The question this article seeks to answer is whether for $m \gg 0$, it holds that $dim_k\Gamma(X_k, mK_{X_k}) = dim_K\Gamma(X_K, mK_{X_K})$, and if more generally, the same holds for Kawamata log-terminal pairs. As far as the author is aware, the only existing results in this direction are the following result due to Junescue Suh, which uses the techniques of [KU], as well as a W_2-lifting hypothesis in place of the Kawamata-Viehweg Vanishing theorem for characteristic 0, and a result due to Tanaka, which assumes a certain ample divisor is added to the pair. Suh's theorem is:

Theorem 2. [Suh, 1.2.1(ii), 1.2.4] *Let R be a discrete valuation ring whose fraction field K (resp. residue field k) has characteristic zero (resp. is perfect of characteristic $p > 0$) and let X/R be a proper smooth algebraic space of relative dimension 2. If X_k lifts to $W_2(k)$ and is of general type, then one has*

$$P_m(X_K) = P_m(X_k)$$

for every integer $m \geq 2$. If moreover X_k has reduced picard scheme then $P_m(X_K) = P_m(X_k)$ for all $m \geq 1$.

Tanaka's theorem is:

Theorem 3. [Tan3, 7.3] *Let X be a smooth projective threefold. Let S be a smooth prime divisor on X and let A be an ample $\mathbb{Z}$-divisor on X such that*

(1) $K_X + S + A$ is nef, and

(2) $\kappa(S, K_S + A|_S) \neq 0$.

Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that, for every integer $m \geq m_0$, the natural restriction map

$$H^0(X, m(K_X + S + A)) \to H^0(S, m(K_S + A|_S))$$

is surjective.

The proof of the above theorem uses some interesting trace of Frobenius methods which are quite different from the techniques used in this article. In this article, minimal model techniques, combined with [KU] are used to gain a result similar to the above theorems but without the $W_2(k)$ lifting hypothesis or the ample $\mathbb{Z}$-divisor A, and with the added benefit that it holds for Kawamata log-terminal pairs. The main theorem is the following.

Theorem 4. *Let (X, Δ) be a kawamata log-terminal pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 2$ and perfect fraction field K. Assume that (X, Δ) is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists a computable m_0 (depending on the intersection numbers of K_X and Δ on the fibers) such that for all $m \in \mathbb{Z}^+$ with $m_0|m$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

If $\Delta = 0$ then $m_0 = 2$.

Consequences of the above theorem are finite generation of the canonical ring, the ability to run a minimal model program in mixed characteristic over a DVR in relative dimension 2, as well as the invariance of the log Kodaira dimensions of Kawamata log-terminal pairs in this setting.

Proposition 5. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k and perfect fraction field K. Assume that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over R. Then the canonical ring Proj $R(K_X + \Delta)$ is finitely generated and a contraction of a $K_{X_k} + \Delta_k$-negative extremal ray cut out by an* *ample H on R induces a contraction of a $K_X + \Delta$ negative extremal ray.*

Corollary 6. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with residue field k and fraction field K such that k is algebraically closed of characteristic $p > 0$. Assume $K_X + \Delta$ is pseudo-effective and simple normal crossings over R. Then the numerical Kodaira dimension satisfy*

$$\nu\left(K_{X_K} + \Delta_K\right) = \nu\left(K_{X_k} + \Delta_k\right).$$

As a result of abundance for log surfaces, the log Kodaira dimension also satisfy

$$\kappa\left(K_{X_K} + \Delta_K\right) = \kappa\left(K_{X_k} + \Delta_k\right).$$

Thanks to my advisor, Professor Hacon, for helping catch bugs in early versions of this paper.

2. Background

Following the notation of [Suh], let p be a prime number and let R be a discrete valuation ring with residue field k and fraction field K, such that K is either of characteristic 0 or p. For a scheme Z over R, Z_k will denote the special fiber $Z \otimes_R k$ and Z_K will denote the generic fiber $Z \otimes_R K$. First a few facts about discrete valuation rings which will help to run the minimal model program over a discrete valuation ring.

The following lemma of Katsura and Ueno is a main ingredient to theorem 4. The proof is included almost verbatim except for noting that the residue field is unchanged after the extension in the proof.

Lemma 7. [KU, 9.4] *Let $\varphi : X \to Spec(R)$ be an algebraic two-dimensional space, proper, separated, and of finite type over $spec(R)$, where R is a DVR with algebraically closed residue field k, and field of fractions K. Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ. If X_k contains an exceptional curve of the first kind e, there exists a DVR $\tilde{R} \supset R$, with residue field isomorphic to k, and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \to \tilde{Y}$ over $Spec\left(\tilde{R}\right)$ such that on the closed fibre, π induces the contraction of the exceptional curve e. Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.*

Proof. By [A1, Cor 6.2] $Hilb_{X/Spec(R)}$ is represented by an algebraic space $\mathscr{H}$ which is locally of finite type over $Spec(R)$. Let Y be the irreducible component containing the point $\{e\}$ corresponding to the exceptional curve e on the special fiber. Then $e \approx \mathbb{P}^1_k$ and $N_{e/X} \approx \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, so Y is regular at $\{e\}$ and of dimension 1. Since e is fixed in the special fiber, the structure morphism $p : Y \to Spec(R)$ is surjective. By lemma 38 and c.f. remark 39, we can find an étale cover $\tilde{R} \supset R$ and a morphism $j : Spec\left(\tilde{R}\right) \to Y$ over $Spec(R)$ with $j(\tilde{o}) = \{e\}$ (if R is not already complete, then first extend R to a complete DVR using [Gr, 0.6.8.2,3] so that $\tilde{R}$ is again a DVR). As k

is assumed algebraically closed, and $\tilde{R} \to R$ is unramified, then the extension of residue fields is finite and separable at the closed point of R, and hence an isomorphism of residue fields. Let $\hat{p} : \mathscr{E} \to Spec\left(\tilde{R}\right)$ be the pull-back of the universal family over Y. As the closed fibre $\mathscr{E}_0$ is a projective line, we may choose the morphism j in such a way that the generic fibre $\mathscr{E}_1$ of $\hat{p}$ is also a projective line. Moreover, $\mathscr{E}$ can be considered as a smooth closed algebraic subspace of codimension 1 in $\hat{X} = X \otimes \tilde{R}$. By lemma 8, $-1 = \mathscr{E}_0^2 = \mathscr{E}_1^2$. Hence $\mathscr{E}_1$ is also an exceptional curve of the first kind. Hence by [A2, Cor 6.11] there exists a contraction morphism $\pi : \hat{X} \to \hat{X}$ over $Spec\left(\tilde{R}\right)$ which contracts $\mathscr{E}$ to a section of $\tilde{\varphi} : \tilde{X} \to Spec\,(R)$ where $\tilde{\varphi}$ is proper, smooth, separated and of finite type over $Spec\left(\tilde{R}\right)$. □

Lemma 8. [KU, 9.3] *In the situation of the above Lemma, if D, D' are divisors on X, then*

$$K_{X_k}^2 = K_{X_K}^2$$
$$\left(K_{X_k} \cdot D_k\right)_{X_k} = \left(K_{X_K} \cdot D_K\right)_{X_K}$$
$$\left(D_k \cdot D'_k\right)_{X_k} = \left(D_K \cdot D'_K\right)_{X_K}.$$

Definition 9. [Ko, 1.7] Let X be a scheme. Let $p \in X$ be a regular point with ideal sheaf $\mathfrak{m}_p$ and residue field $k\,(p)$. Then $x_1, ..., x_n \in \mathfrak{m}_p$ are called local coordinates if their residue classes $\overline{x_1}, ..., \overline{x_n}$ form a $k\,(p)$-basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Let $D = \sum a_i D_i$ be a Weil divisor on X. We say that (X, D) has simple normal crossings or snc at a point $p \in X$ if X is regular at p and there is an open neighborhood $p \in X_p \subset X$ with local coordinates

$x_1, ..., x_n \in \mathfrak{m}_p$ such that $X_p \cap Supp\ D \subset (x_1 \cdot \cdots \cdot x_n = 0)$. We say that (X, D) is snc if it is snc at every point.

We say that (X, D) has normal crossing or nc at a point $p \in X$ if $\left(\hat{X}_K, D|_{\hat{X}_K}\right)$ is snc at p where $\hat{X}_K$ denotes the completion at p and K is a separable closure of $k(p)$. We say that (X, D) is nc if it is nc at every point.

If (X, D) is defined over a perfect field, this concept is also called log smooth.

The following statement of resolution of singularities will be used.

Theorem 10. [CP2014, 1.1] *Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*

(i) X' is everywhere regular

(ii) π induces an isomorphism $\pi^{-1}(Reg\ X) \approx Reg\ X$

(iii) $\pi^{-1}(Sing\ X)$ is a strict normal crossings divisor on X'.

Definition 11. A log resolution of a pair (X, Δ) is a proper birational morphism $f : Y \to X$ from a regular variety such that the exceptional locus $Ex(f)$ is a divisor and $f^{-1}(\Delta) \cup Exc(f)$ has simple normal crossings support.

Definition 12. Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair (X, Δ) has terminal (respectively Kawamata log-terminal i.e. "klt", respectively log-canonical) singularities if for any log resolution $f : Y \to X$ of (X, Δ) such that E_i are exceptional curves on Y, then

$$K_Y + \Delta_Y = f^* (K_X + \Delta) + \sum a_i E_i$$

where $a_j > 0$ (respectively $a_j > -1$, respectively $a_j \geq -1$) and Δ_Y is the strict transform of Δ.

The following ingredients of the minimal model program for surfaces in positive characteristic are used [KK94], [Tan].

Definition 13. Let $f : X \to Z$ be a projective birational morphism of algebraic spaces such that $f_* \mathcal{O}_X = \mathcal{O}_Z$ and $dim\ NE(X/Z) = 1$ and f contracts some divisor. Then f is called a divisorial contraction. If instead f contracts some subvariety of codimension ≥ 2 and no divisors, then f is called a small contraction. If f is a small (resp. divisorial) contraction and $-D$ is $\mathbb{R}$-Cartier and relatively ample, then f is a D-flipping contraction (resp. D-divisorial contraction).

Lemma 14. [KK94, 2.3.5] *Let (S, B) be a log-canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C.(K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

Theorem 15. [Tan, 4.4] *Let X be a projective normal surface and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let H be an $\mathbb{R}$-cartier ample $\mathbb{R}$-divisor. Then the following assertions hold:*

(1) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$

(2) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i]$

(3) Each C_i in (1) and (2) is rational or $C_i = B_j$ for some B_j with $B_j^2 < 0$.

Theorem 16. [Tan, 5.3] *Let X be a projective normal surface and let C be a curve in X such that $r(K_X + C)$ is Cartier for some positive integer r.*

(1) If $C \cdot (K_X + C) < 0$, then $C \approx \mathbb{P}^1$.

(2) If $C \cdot (K_X + C) = 0$, then $C \approx \mathbb{P}^1$ or $\mathcal{O}_C\left((K_X + C)^{[r]}\right) \approx \mathcal{O}_C$.

Definition 17. [HK, 5.E] If $K_X + \Delta$ is an effective KLT pair in dimension 2, then for any ample divisor H', we can find $h \in \mathbb{R}_{>0}$ and an $\mathbb{R}$-divisor $H \sim_{\mathbb{R}} hH'$ such that $(X, \Delta + H)$ is klt and $K_X + \Delta + H$ is nef and big. Let

$$\lambda = inf\{t \geq 0 \mid K_X + \Delta + tH \text{ } is \text{ } nef\}.$$

If $\lambda = 0$, then $K_X + \Delta$ is nef, and thus (X, Δ) is minimal. If $\lambda > 0$, then, by Theorem 15, there exists a $(K_X + \Delta)$-negative extremal ray R such that $(K_X + \Delta + \lambda H).R = 0$. Then, c.f. [KK94, 2.3], if the corresponding contraction $\phi : X \to X'$ does not result in a log Del

Pezzo surface or a birationally ruled surface, then setting $H' = \phi_* H$ and $\Delta' = \phi_* \Delta$, the divisor $K_{X'} + \Delta' + \delta H'$ is nef. Then the process may be repeated. The process either terminates at some step in a log minimal model or one of the aforementioned surfaces since there are no flips in dimension 2. The end result is a finite sequence of real numbers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ such that $K_{X_n} + \Delta_n + \lambda_n H_n$ is nef and $X \to X_n$ is a minimal model for $(X, \Delta + \lambda_n H)$.

The proofs in this article will also make use of several vanishing theorems.

Theorem 18. *(Kawamata-Viehweg Vanishing* [Tan2, 2.11]*) Let* (X, Δ) *be a projective klt surface (over an algebraically closed field of characteristic* $p > 0$*) where* Δ *is an effective* $\mathbb{R}$*-divisor. Let* N *be a nef* $\mathbb{R}$*-cartier* $\mathbb{R}$*-divisor with* $\nu(X, N) \geq 1$*. Let* D *be a* $\mathbb{Q}$*-Cartier* $\mathbb{Z}$*-divisor such that* $D - (K_X - \Delta)$ *is nef and big. Then there exists a positive real number* $r(\Delta, D, N)$ *such that* $H^i(X, \mathcal{O}_X(D + rN + N')) = 0$ *for every* $i > 0$*, every positive real number* $r \geq r(\Delta, D, N)$*, and every nef* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* N' *such that* $rN + N'$ *is a Cartier divisor.*

Theorem 19. [Ek, II.6] *Let* X *be a minimal surface of general type and let* $\mathcal{L}$ *be an invertible sheaf that is numerically equivalent to* $\omega_S^{\otimes i}$ *for some* $i \geq 1$*. Then* $H^1(X, \mathcal{L}^{\otimes 2}) = 0$ *except possibly for certain surfaces in characteristic* 2 *with* $\chi(\mathcal{O}_X) \leq 1$ *.*

Theorem 20. [Ter] *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X. Assume that either*

(1) $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or

(2) X is of general type with

$p \geq 3$ and $(D^2) > vol(X)$ or

$p = 2$ and $(D^2) > max\{vol(X), vol(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all $i > 0$.

The following properties of base change will be used:

Theorem 21. [Har, III.10.2] *Let $f : X \to Y$ be a morphism of schemes of finite type over a field k. Then f is smooth of relative dimension n if and only if :*

(1) f is flat; and

(2) for each point $y \in Y$, let $X_{\overline{y}} = X_y \otimes_{k(y)} k(y)^-$ where $k(y)^-$ is the algebraic closure of $k(y)$. Then $X_{\overline{y}}$ is equidimensional of dimension n and regular. (We say the "fibres of f are geometrically regular of equidimension n.")

Theorem 22. [GW, 6.28] *Let k be a field, X a k-scheme locally of finite type, and let $x \in X$ be a closed point. Let $d \geq 0$ be an integer. We fix an algebraically closed extension K of k, and write $X_K = X \otimes_k K$. The following are equivalent:*

(i) The k-scheme X is smooth of relative dimension d at x.

(ii) For every point $\overline{x} \in X_K$ lying over x, X_K is smooth of relative dimension d at $\overline{x}$.

(iii) For every point $\overline{x} \in X_K$ lying over x, the completed local ring $\hat{\mathcal{O}}_{X_K,\overline{x}}$ is isomorphic to a ring of formal power series $K[[T_1, ..., T_d]]$ over K.

(iv) For every point $\overline{x} \in X_K$ lying over x, the local ring $\mathcal{O}_{X_K,\overline{x}}$ is regular and has dimension d.

(v) The equalities $dim_{\kappa(x)} T_x (X/k) = dim \mathcal{O}_{X,x} = d$ hold.

If these conditions are satisfied, then

(vi) The local ring $\mathcal{O}_{X,x}$ is regular and has dimension d.

Furthermore, if $\kappa(x) = k$, then the final condition implies the other ones.

As a consequence of the above theorems,

Proposition 23. *Let (X, Δ) be a terminal (resp. klt) pair which is simple normal crossings over a DVR R with perfect residue field k and perfect fraction field K. Let (X', Δ') denote the pair on the base change to a complete DVR R' with algebraically closed residue field. Then (X'_k, Δ'_k) and (X'_K, Δ'_K) are log-smooth and terminal (resp. klt.)*

Proof. By adjunction (X_k, Δ_k) and (X_K, Δ_K) are terminal (resp. klt) and snc by definition 9. Since smoothness is preserved by base-change, then all strata of (X', Δ') are smooth. Also, (X'_k, Δ'_k) is by definition log smooth after base change to the algebraic closure of k, since the algebraic closure of a perfect field is equal to its separable closure. Thus (X', Δ') is log-smooth.

Next, since k is assumed perfect, the notions of smooth and regular coincide. Thus by Theorems 22, and 21 we have that the base change $X \to X'$ over the algebraic closure k' is smooth of relative dimension 0, hence an étale base change. Thus by [Ko, 2.14,2.15], (X_k, Δ_k) and (X_K, Δ_K) are terminal (resp. klt) iff (X'_k, Δ'_k) and (X'_K, Δ'_K) are terminal (resp. klt). $\square$

Definition 24. [Laz] Let D be a pseudoeffective $\mathbb{R}$-divisor on a normal projective variety X. The diminished base locus is defined as

$$\mathbb{B}_{-}(D) := \bigcup_{\substack{A \ \ Ample \\ D+A \ \ \mathbb{Q}-Cartter}} \mathbb{B}(D+A)$$

where $\mathbb{B}(D+A) = \bigcap_{n \geq 1} Bs(n(D+A))$ is the stable base locus.

Definition 25. Let D be a big Cartier divisor. Let F_m be the fixed divisor $|mD|_{fix}$. Then $F_{m+n} \leq F_m + F_n$ and the limit

$$N_\sigma(D) := \lim_{m \to \infty} \frac{1}{m} F_m$$

exists as an $\mathbb{R}$-divisor. We have $mN_\sigma(D) \leq F_m$ and for the $\mathbb{R}$-divisor $P_\sigma(D) := D - N_\sigma(D)$, we have an isomorphism

$$H^0(X, \mathcal{O}_X(mD)) \approx H^0(X, \mathcal{O}_X(\llcorner mP_\sigma(D) \lrcorner))$$

for any $m > 0$. The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the sectional decomposition.

Remark 26. If D is not big, then $|mD|$ may be empty for certain positive integers m, and thus, in defining $N_\sigma(D)$, it is necessary to consider only the semigroup $\mathbb{N}(D)$ of m such that $|mD| \neq \emptyset$ for such D. In this article, only the sectional decomposition of big divisors is considered.

3. Effective KLT Kawamata Viehweg Vanishing

In this modification I note that, at least in the smooth case, a simple modification of Tanaka's theorem 3 gives you version of the theorem in which the multiple of the nef divisor is at least computable.

Theorem 27. *(Effective KLT Kawamata-Viehweg Vanishing) Let (X, Δ) be a normal projective klt surface (over an algebraically closed field of characteristic $p > 2$) where Δ is an effective $\mathbb{R}$-divisor. Let N be a nef and big $\mathbb{R}$-cartier $\mathbb{R}$-divisor with $N^2 > vol(X)$. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D - (K_X - \Delta)$ is nef and big. Then $H^i(X, \mathcal{O}_X(D + rN + N')) = 0$ for every $i > 0$, every positive real number $r \geq 1$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor N' such that $rN + N'$ is a Cartier divisor.*

Proof. Verbatim from [Tan2, 2.11] except that Weak Kodaira Vanishing [Tan2, 2.4] in the proof of [Tan2, 2.6] is replaced by Terakawa's theorem 20. □

4. Proof of Main Result

The proof of the main result goes in several steps. The first step is to show that under more restrictive assumptions on the pair (X, Δ), that the invariance of plurigenera holds. The following is similar to [Tan3, 7.3] but without an ample boundary. The proof is similar to the Theorem [Suh, 1.2.1(ii)], where empty boundary is considered and there is, in addition, assumed a W_2 lifting hypothesis.

Proposition 28. *Let (X, Δ) be a terminal log smooth pair of relative dimension 2 over a DVR R with algebraically closed residue field k of characteristic $p > 2$ and perfect fraction field K. Assume that $K_X + \Delta$ is big and $\mathbb{Q}$-Cartier. Assume that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$. Then there exists an m_0 such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

The proof of proposition 28, is given in the following two claims.

Claim. Assumptions as above, after passing to an extension R' of R, there is a proper, smooth algebraic space X^{min}/R' and an R' morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef.

Proof. As k is algebraically closed of characteristic $p > 0$, then by the Cone Theorem 15,

$$\overline{NE}(X_k) = \overline{NE}(X_k)_{K_{X_k}+\Delta_k \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

with each C_i is rational or $C_i = B_j$ for some B_j a component of Δ with $B_j^2 < 0$. Under the assumption that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$, then actually each C_i is rational and is not a component of Δ_k. Thus $C_i.\Delta_k \geq 0$ so $C_i.K_{X_k} < 0$, and since $N_\sigma(K_{X_k} + \Delta_k)$ is a $\mathbb{Q}$-divisor by [Tan, 7.1], then Theorem 16 implies $C_i \approx \mathbb{P}^1$. By Lemma 14, C_i can be contracted to a log-canonical point, which is actually a smooth point under the terminal assumption. Thus C_i is an exceptional curve of the first kind, so it is possible to apply Lemma 7.

By Lemma 7, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \to \tilde{X}$ induces the contraction of C_i on X_k and X_K, and further $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since k is algebraically closed, it induces identity on residue fields. Now I need to work with $\tilde{X}$. In order to apply the Cone Theorem 15 again, I need $\tilde{X}_k$ to be projective, but a smooth algebraic space of dimension 2, proper, separated, and of finite type over an algebraically closed field is projective, so $\tilde{X}_k$ is projective. Thus the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no problem in extending R. As the Picard number drops each step, there are only finitely many steps. □

Claim 29. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef.

Proof. This is a slight modification of [KU, 9.6]. By Lemma 7, it is impossible to reach a minimal X_K before a minimal X_k. Thus assume that $K_{X_k} + \Delta_k$ is nef but $K_{X_K} + \Delta_K$ is not nef. Let X^* denote the minimal model for X_K. Then letting $K_{X_K}+\Delta_K = f^*(K_{X^*} + \Delta_{X^*})+E$, then, as (X_K, Δ_K) is terminal, $(K_{X^*} + \Delta_{X^*})^2 > (K_{X_K} + \Delta_K)^2$.

For sufficiently large m such that $m(K_{X_k} + \Delta_k)$ is Cartier, then $(m-1)K_{X_k} + m\Delta_k$ is also Cartier.

$$h^0(m(K_{X_K} + \Delta_K)) = h^0(m(K_{X^*} + \Delta_{X^*})).$$

Since $K_{X_k}+\Delta_{X_k}$ is nef and big, then if m is such that $((m-1)(K_{X_k} + \Delta_{X_k}))^2 > vol(X_k)$ with $m(K_{X_k} + \Delta_{X_k})$ also Cartier, the higher cohomologies $H^i(m(K_{X_k} + \Delta_{X_k}))$, $i > 0$ vanish by Theorem 27. (The same thing happens on X_K by semi-continuity.) Thus, by the Riemann-Roch formula (see [Liu, Chapt. 9.1], [Har, III.7], [HS, A.4.2.1] and [BC, Cor. 9] for the necessary ingredients when K is perfect instead of algebraically closed) we have:

$$h^0(m(K_{X_K} + \Delta_K)) = h^0(m(K_{X^*} + \Delta_{X^*}))$$

$$= \frac{1}{2}m^2(K_{X^*} + \Delta_{X^*})^2 - \frac{m}{2}K_{X^*}.(K_{X^*} + \Delta_{X^*}) + \chi(\mathcal{O}_{X_K}).$$

Similarly:

$$h^0(m(K_{X_k} + \Delta_k)) = \frac{1}{2}m^2(K_{X_k} + \Delta_{X_k})^2 - \frac{m}{2}K_{X_k}.(K_{X_k} + \Delta_k) + \chi(\mathcal{O}_{X_k}).$$

So now there is a contradiction to semicontinuity in that, for some computable m (given the intersection numbers),

$$h^0(m(K_{X_K} + \Delta_K)) > h^0(m(K_{X_k} + \Delta_k)).$$

□

Now, by the above claim, there is a proper smooth algebraic space X^{min} such that X_k^{min} and X_K^{min} are obtained by blowing down -1 curves, and are minimal, proper, and smooth. Hence, by the Kawamata Viehweg vanishing (either Theorem 18, or the usual version in characteristic 0, depending on whether X has equal or mixed characteristic), there exists computable $m_0 \gg 0$ such that for $m > m_0$,

$$h^i\left(m\left(K_{X_K} + \Delta_K\right)\right) = h^i\left(m\left(K_{X_k} + \Delta_k\right)\right) = 0.$$

In fact, if $\Delta = 0$, then Ekedahl's vanishing Theorem 19 can be applied.

Thus, using the same Euler-characteristic argument as in the above claim, and using semicontinuity, it follows that for $m > m_0$,

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

The next step is to remove the restriction on the base locus from proposition 28.

Corollary 30. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists a computable m_0 depending on the intersection numbers, such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right)$$

thus, after possibly extending R, the canonical ring $R(K_X+\Delta)$ is finitely generated over R

Proof. Since the hypothesis and conclusion are preserved by base change, then extending R, we may assume that k is algebraically closed and that R is complete. The proof follows similarly to [HMX2010, 1.6] except that Proposition 28 is used in place of Kawamata Viehweg Vanishing. Recall that the log-canonical ring of $K_{X_k}+\Delta_k$ is finitely generated (c.f. [Tan, 7.1]) so $N_\sigma(K_{X_k}+\Delta_k)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma(K_{X_k}+\Delta_k)$$

and let $0 \leq \Theta \leq \Delta$ be a $\mathbb{Q}$-divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. Replacing (X,Δ) by a blow-up, assume (X,Δ) is terminal. By definition of N_σ, letting $m \gg 0$ to fit the hypothesis of Proposition 28, and sufficiently divisible such that $m(K_{X_k}+\Theta_k)$ is integral, then

$$h^0(m(K_{X_k}+\Delta_k)) = h^0(m(K_{X_k}+\Theta_k))$$

and furthermore, by Proposition 28,

$$h^0(m(K_{X_k}+\Theta_k)) = h^0(m(K_{X_K}+\Theta_K)).$$

As $\Theta \leq \Delta$, the first part of the theorem follows by semicontinuity.

Now by the theory of surfaces, there is $m \gg 0$, such that $\bigoplus_{m\geq 0} H^0(ml(K_{X_k}+\Delta_k))$ is generated in degree 1, so that there is a surjection

$$S^l H^0(m(K_{X_k}+\Delta_k)) \twoheadrightarrow H^0(ml(K_{X_k}+\Delta_k)).$$

Consider the following diagram:

$$\begin{array}{ccc} S^l H^0\left(m\left(K_X+\Delta\right)/R\right) & \xrightarrow{\alpha} & H^0\left(ml\left(K_X+\Delta\right)/R\right) \\ \downarrow & & \downarrow \\ S^l H^0\left(m\left(K_{X_k}+\Delta_k\right)\right) & \xrightarrow{\twoheadrightarrow} & H^0\left(ml\left(K_{X_k}+\Delta_k\right)\right) \end{array}$$

The vertical maps are surjective by the first part of the theorem (and applying Nakayama's lemma to pull back the generators of the k-modules). Now again using, Nakayama's Lemma [GH, Chapter 5.3], surjectivity of the bottom map implies surjectivity of α, and thus $R\left(K_X+\Delta\right)$ is finitely generated over R. □

Remark 31. My advisor claims that finite generation of $R\left(K_X+\Delta+\epsilon A\right)$ for any ϵ implies the minimal model exists. This may be true, but I am not well versed enough in the minimal model theory to understand the proof yet. It be explored in the sequel.

The following is another Corollary of Theorem 28, for pairs not of general type. This is a result of the proof of Theorem 28 not being affected by adding an ample divisor H. By abundance for log surfaces, this gives a log version of [KU].

Corollary 32. *Let (X, Δ) be a klt log-smooth pair of relative dimension 2 over a DVR R with residue field k and fraction field K such that k is algebraically closed of characteristic $p > 0$. Assume $K_X + \Delta$ is pseudo-effective. Then the numerical kodaira dimension satisfy*

$$\nu\left(K_{X_K}+\Delta_K\right)=\nu\left(K_{X_k}+\Delta_k\right).$$

As a result of abundance for log surfaces, the log kodaira dimension also satisfy

$$\kappa\left(K_{X_K}+\Delta_K\right)=\kappa\left(K_{X_k}+\Delta_k\right).$$

Proof. For any ample A, there exists an m_0 such that for $m > m_0$, $K_X+\Delta+\frac{1}{m}A = K_X+\Delta'$ is klt and and big (since $K_X+\Delta$ is assumed pseudo-effective). Thus the result follows from Corollary 30. □

5. Supplementary Algebra Results

5.1. Discrete Valuation Rings.

This section contains some basic results on discrete valuation rings useful for understanding Lemma 7.

Definition 33. A discrete valuation ring (DVR for short) R is an integral domain which is an integrally closed noetherian local ring with Krull dimension one.

Lemma 34. *Let A be a discrete valuation ring with fraction field K. Let L/K be any finite separable extension, and let B denote the integral closure of A in L. Then B is a finite A-module, and if A is complete, then so is B, and B is a DVR. That is, finite extensions are complete and unsplit.*

Definition 35. Let A, B be Noetherian local rings. A local homomorphism $A \to B$ is said to be an unramified homomorphism of local rings if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,

(2) $\kappa(\mathfrak{m}_B)$ is a finite separable extension of $\kappa(\mathfrak{m}_A)$, and

(3) B is essentially of finite type over A (i.e. B is the localization of a finite type A-algebra at a prime.

Definition 36. Let A, B be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be an étale homomorphism of local rings if it is a flat and unramified homomorphism of local rings. If Y is a locally Noetherian scheme, and $f : X \to Y$ is a morphism of schemes which is locally of finite type, then f is said to be étale if it is étale at all its points.

Lemma 37. [Deb, 6.14] *Let A be a noetherian integrally closed local ring with fraction field K and set $S = Spec\,(A)$. Let $\phi : X \to S$ be an étale cover. Then X is also normal, and in particular, it can be written as the coproduct of its (finitely many) irreducible components. Furthermore, given a connected component X_0 of X, the induced étale cover $X_0 \to S$ is the normalization of S in $k\,(S) \hookrightarrow k\,(X_0)$.*

From the above lemmas, it is clear that an étale cover of a complete DVR results is a complete DVR.

Lemma 38. [Bo, 2.2.14] *Let $f : X \to S$ be a smooth morphism (of schemes). Let s be a point of S, and let x be a closed point of the fibre $X_s = X \times_S Spec\; k\,(s)$ such that $k\,(x)$ is a separable extension of $k\,(s)$. Then there exists an étale morphism $g : S' \to S$ and a point $s' \in S'$ above s such that the morphism $f' : X \times_S S' \to S'$ obtained from f by* *the base change $S' \to S$ admits a section $h : S' \to X \times_S S'$, where $h(s')$ lies above x, and where $k(s') = k(x)$.*

Remark 39. It seems to me, that if k is perfect, the proof of the above lemma goes through when $f : X \to S$ is merely smooth at $x \in X$ and $f : X \to S$ is locally of finite type.

5.2. **Riemann-Roch for Perfect Fields.** The standard Riemann-Roch theorem for algebraic surfaces is stated over an algebraically closed field. I'm sure the version for perfect fields is known if true, but since I was unable to find a reference, in this section I give necessary ingredients to extending the proof in [Har, Chapter 5.1] to perfect fields.

Lemma 40. [Liu, 7.3.16] *Let X be a projective variety over a field k. Let*

$$0 \to F \to G \to H \to 0$$

be an exact sequence of coherent sheaves on X. Then

$$\chi(G) = \chi(F) + \chi(H).$$

Theorem 41. [Liu, 7.3.17] *Let X be a projective curve over a field k. Let D be a Cartier divisor on X. Then we have*

$$\chi(\mathcal{O}_X(D)) = deg\ D + \chi(\mathcal{O}_X).$$

Definition 42. [Liu, 4.1.2, 8.3.1]A normal Noetherian integral domain of dimension 0 or 1 is called a Dedekind domain. A normal locally Noetherian scheme of dimension 0 or 1 is called a Dedekind scheme. Let S be a Dedekind Scheme. We call an integral, projective, flat S-scheme $\pi : X \to S$ of dimension 2 a fibered surface over S. If $dim\ S = 0$ then X is an integral, projective, algebraic surface over a field. An irreducible Weil divisor D is called horizontal if $dim\ S = 1$ and if $\pi|_D : D \to S$ is surjective. If $\pi(D)$ is reduced to a point, we say that D is vertical.

Theorem 43. [Liu, 9.1.12]*(Intersection on a fibered surface). Let $X \to S$ be a regular fibered surface. Let $s \in S$ be a closed point. Then there exists a unique bilinear map (of $\mathbb{Z}$-modules)*

$$i_s : Div(X) \times Div_s(X) \to \mathbb{Z}$$

which verifies the following properties:

(a) If $D \in Div(X)$ and $E \in Div_s(X)$ have no common component, then

$$i_s(D, E) = \sum_x i_x(D, E)\,[k(x) : k(s)],$$

where x runs through the closed points of X_s.

(b) The restriction of i_s to $Div_s(X) \times Div_s(X)$ is symmetric.

(c) $i_s(D, E) = i_s(D', E)$ if $D \sim D'$.

(d) If $0 < E \leq X_s$, then

$$i_s(D, E) = deg_{k(s)}\mathcal{O}_X(D)|_E.$$

Theorem 44. [Liu, 9.1.37] *Let $X \to S$ be a regular fibered surface, $s \in S$ a closed point, and $E \in Div_s(X)$ such that $0 < E < X_s$ (the second inequality is an empty condition if dim $S = 0$). Then we have*

$$\omega_{E/k(s)} \approx \left(\mathcal{O}_X(E) \otimes \omega_{X/S}\right)|_E,$$

and if $K_{X/S}$ is a canonical divisor,

$$p_a(E) = 1 + \frac{1}{2}\left(E^2 + K_{X/S} \cdot E\right).$$

Theorem 45. [Har, III.7.7] *Let X be a projective Cohen-Macaulay scheme of equidimension n over a field k. Then for any locally free sheaf $\mathscr{F}$ on X there are natural isomorphisms*

$$H^i(X, \mathscr{F}) \approx H^{n-i}\left(X, \mathscr{F}^\vee \otimes \omega_X^\circ\right)'$$

where ω_X° is the dualizing sheaf on X.

Theorem 46. [BC, Cor 9.] *Let X be a nonsingular projective variety of dimension n over a perfect field k. Then $\omega_X^\circ = \Omega_X^n$.*

Finally note that the following Riemann-Roch formula holds over perfect fields. The proof is the same as the one in [Har] Chapter 5, but the field is no longer assumed algebraically closed. All the necessary results are stated above.

Theorem 47. *Let D be a divisor on a nonsingular, projective surface X over a perfect field k. Then*

$$\chi\left(\mathcal{O}_X\left(D\right)\right) = \frac{1}{2}D.\left(D-K\right) + \chi\left(\mathcal{O}_X\right).$$

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