INVARIANCE OF PLURIGENERA FOR LOG SURFACES OVER A DVR IN MIXED CHARACTERISTIC

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ABSTRACT. Log deformation invariance of plurigenera, existence of minimal models, and finite generation of the canonical ring for a family of two dimensional varieties over a DVR in positive or mixed characteristic.

CONTENTS

1. INTRODUCTION

The purpose of this article is to consider the characteristic $p > 0$ analogue of the famous theorem of Siu in the case of an algebraic variety X/R of relative dimension 2 over a DVR . Siu's theorem states

Theorem 1. [Siu] *Let $\pi : X \to \Delta$ be a smooth projective family of compact complex manifolds parametrized by the open unit 1-disk Δ. Assume that the family $\pi : X \to \Delta$ is of general type. Then for every* *positive integer m, the plurigenus $dim_{\mathbb{C}}\Gamma\left(X_t, mK_{X_t}\right)$ is independent of $t \in \Delta$, where $X_t = \pi^{-1}\left(t\right)$ and K_{X_t} is the canonical line bundle of X_t.*

The above theorem in characteristic 0 has been generalized to all Kodaira dimensions and to kawamata log-terminal, log smooth pairs c.f. [Siu2], [HMX2010, 1.8], and [BP].

In positive characteristic, the above theorem in general does not hold even when the fibers are surfaces with kodaira dimension 1, since there are examples of Katsura and Ueno [KU] where a fiber with wild ramification causes the geometric genus to jump rather than being constant in the family. Similarly, Suh [Suh] has constructed counterexamples in Kodaira dimension 2 to invariance of the geometric genus.

However, in their paper, Katsura and Ueno also show that in the case of a smooth algebraic variety X/R over a DVR of relative dimension 2 with residue field k and fraction field K, then $\kappa\left(X_k\right) = \kappa\left(X_K\right)$. So the question becomes whether some asymptotic version of Siu's theorem holds in this case. The question this article seeks to answer is whether for $m \gg 0$, it holds that $dim_k\Gamma\left(X_k, mK_{X_k}\right) = dim_K\Gamma\left(X_K, mK_{X_K}\right)$, and if more generally, the same holds for kawamata log-terminal pairs. As far as the author is aware, the only existing results in this direction are the following result due to Junescue Suh, which uses the techniques of [KU], as well as a W_2-lifting hypothesis in place of the Kawamata-Viehweg Vanishing theorem for characteristic 0, and a result due to Tanaka, which assumes a certain ample divisor is added to the pair. Suh's theorem is:

Theorem 2. [Suh, 1.2.1(ii), 1.2.4] *Let R be a discrete valuation ring whose fraction field K (resp. residue field k) has characteristic zero (resp. is perfect of characteristic $p > 0$) and let X/R be a proper smooth algebraic space of relative dimension 2. If X_k lifts to $W_2(k)$ and is of general type, then one has*

$$P_m(X_K) = P_m(X_k)$$

for every integer $m \geq 2$. If moreover X_k has reduced picard scheme then $P_m(X_K) = P_m(X_k)$ for all $m \geq 1$.

Tanaka's theorem is:

Theorem 3. [Tan3, 7.3] *Let X be a smooth projective threefold. Let S be a smooth prime divisor on X and let A be an ample $\mathbb{Z}$-divisor on X such that*

(1) $K_X + S + A$ is nef, and

(2) $\kappa(S, K_S + A|_S) \neq 0$.

Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that, for every integer $m \geq m_0$, the natural restriction map

$$H^0(X, m(K_X + S + A)) \to H^0(S, m(K_S + A|_S))$$

is surjective.

The proof of the above theorem uses some interesting trace of Frobenius methods which are quite different from the techniques used in this article. In this article, minimal model techniques, combined with [KU] are used to gain a result similar to the above theorems but without the $W_2(k)$ lifting hypothesis or the ample $\mathbb{Z}$-divisor A, and with the added benefit that it holds for Kawamata log-terminal pairs. The main theorem is the following.

Theorem 4. *Let (X, Δ) be a kawamata log-terminal pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 2$ and perfect fraction field K. Assume that (X, Δ) is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists an m_0 such that for all $m \in \mathbb{Z}^+$ with $m_0 | m$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

If $\Delta = 0$ then $m_0 = 2$.

Consequences of the above theorem are the ability to run a minimal model program in mixed characteristic and relative dimension 2, as well as the invariance of the log Kodaira dimensions of Kawamata log-terminal pairs in this setting.

Proposition 5. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k and perfect fraction field K. Assume that $K_X + \Delta$ is a big $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over R. Then the canonical ring Proj $R(K_X + \Delta)$ is finitely generated and a contraction of a $K_{X_k} + \Delta_k$-negative extremal ray cut out by an ample H on R induces a contraction of a $K_X + \Delta$ negative extremal ray.*

Corollary 6. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k and perfect fraction field K. Assume that $K_X + \Delta$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over R. Then the minimal model of (X, Δ) exists over R and a minimal model program can be run until $K_X + \Delta$ is nef on both the generic and special fibers.*

Corollary 7. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with residue field k and fraction field K such that k is algebraically closed of characteristic $p > 0$. Assume $K_X + \Delta$ is pseudo-effective and simple normal crossings over R. Then the numerical Kodaira dimension satisfy*

$$\nu \left(K_{X_K} + \Delta_K\right) = \nu \left(K_{X_k} + \Delta_k\right).$$

As a result of abundance for log surfaces, the log Kodaira dimension also satisfy

$$\kappa \left(K_{X_K} + \Delta_K\right) = \kappa \left(K_{X_k} + \Delta_k\right).$$

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2. Background

Following the notation of [Suh], let p be a prime number and let R be a discrete valuation ring with residue field k and fraction field K, such that K is either of characteristic 0 or p. For a scheme Z over R, Z_k will denote the special fibre $Z \otimes_R k$ and Z_K will denote the generic fiber $Z \otimes_R K$. First a few facts about discrete valuation rings which will help to run the minimal model program over a discrete valuation ring.

The following lemma of Katsura and Ueno is a main ingredient to theorem 4.

Lemma 8. [KU, 9.4] *Let $\varphi : X \to Spec(R)$ be an algebraic space, proper, separated, and of finite type over $spec(R)$, with relative dimension 2, where R is a DVR with algebraically closed residue field k, and field of fractions K. Let X_K (resp. X_k) denote the generic geometric (resp. closed) fibre of φ. If X_k contains an exceptional curve of the first kind e, there exists a DVR $\tilde{R} \supset R$, with residue field isomorphic to k, and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \to \tilde{Y}$ over $Spec\left(\tilde{R}\right)$ such that on the closed fibre, π induces the contraction of the exceptional curve e. Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.*

Lemma 9. [KU, 9.3] *In the situation of the above Lemma, if D, D' are divisors on X, then*

$$K_{X_k}^2 = K_{X_K}^2$$
$$\left(K_{X_k} \cdot D_k\right)_{X_k} = \left(K_{X_K} \cdot D_K\right)_{X_K}$$
$$\left(D_k \cdot D'_k\right)_{X_k} = \left(D_K \cdot D'_K\right)_{X_K}.$$

Definition 10. [Ko, 1.7] Let X be a scheme. Let $p \in X$ be a regular point with ideal sheaf $\mathfrak{m}_p$ and residue field $k(p)$. Then $x_1, ..., x_n \in \mathfrak{m}_p$ are called local coordinates if their residue classes $\overline{x_1}, ..., \overline{x_n}$ form a $k(p)$-basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Let $D = \sum a_i D_i$ be a Weil divisor on X. We say that (X, D) has simple normal crossings or snc at a point $p \in X$ if X is regular at p and there is an open neighborhood $p \in X_p \subset X$ with local coordinates $x_1, ..., x_n \in \mathfrak{m}_p$ such that $X_p \cap Supp\ D \subset (x_1 \cdot \dots \cdot x_n = 0)$. We say that (X, D) is snc if it is snc at every point.

We say that (X, D) has normal crossing or nc at a point $p \in X$ if $\left(\hat{X}_K, D|_{\hat{X}_K}\right)$ is snc at p where $\hat{X}_K$ denotes the completion at p and K is a separable closure of $k(p)$. We say that (X, D) is nc if it is nc at every point.

If (X, D) is defined over a perfect field, this concept is also called log smooth.

The following statement of resolution of singularities will be used.

Theorem 11. [CP2014, 1.1] *Let X be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : X' \to X$ with the following properties:*

(i) X' is everywhere regular

(ii) π induces an isomorphism $\pi^{-1}(Reg\ X) \approx Reg\ X$

(iii) $\pi^{-1}(Sing\ X)$ is a strict normal crossings divisor on X'.

Definition 12. A log resolution of a pair (X, Δ) is a proper birational morphism $f : Y \to X$ from a regular variety such that the exceptional locus $Ex(f)$ is a divisor and $f^{-1}(\Delta) \cup Exc(f)$ has simple normal crossings support.

Definition 13. Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair (X, Δ) has terminal (respectively kawamata log-terminal i.e. "klt", respectively log-canonical) singularities if for any log resolution $f : Y \to X$ of (X, Δ) such that E_i are exceptional curves on Y, then

$$K_Y + \Delta_Y = f^*(K_X + \Delta) + \sum a_i E_i$$

where $a_j > 0$ (respectively $a_j > -1$, respectively $a_j \geq -1$) and Δ_Y is the strict transform of Δ.

The following ingredients of the minimal model program for surfaces in positive characteristic are used [KK94], [Tan].

Definition 14. Let $f : X \to Z$ be a projective birational morphism of algebraic spaces such that $f_*\mathcal{O}_X = \mathcal{O}_Z$ and $dim\ NE(X/Z) = 1$ and f contracts some divisor. Then f is called a divisorial contraction. If instead f contracts some subvariety of codimension ≥ 2 and no divisors, then f is called a small contraction. If f is a small (resp. divisorial) contraction and $-D$ is $\mathbb{R}$-Cartier and relatively ample, then f is a D-flipping contraction (resp. D-divisorial contraction).

Lemma 15. [KK94, 2.3.5] *Let (S, B) be a log canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C.(K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

Theorem 16. [Tan, 4.4] *Let X be a projective normal surface and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let H be an $\mathbb{R}$-cartier ample $\mathbb{R}$-divisor. Then the following assertions hold:*

(1) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$

(2) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i]$

(3) Each C_i in (1) and (2) is rational or $C_i = B_j$ for some B_j with $B_j^2 < 0$.

Theorem 17. [Tan, 5.3] *Let X be a projective normal surface and let C be a curve in X such that $r(K_X + C)$ is Cartier for some positive integer r.*

(1) If $C \cdot (K_X + C) < 0$, then $C \approx \mathbb{P}^1$.

(2) If $C \cdot (K_X + C) = 0$, then $C \approx \mathbb{P}^1$ or $\mathcal{O}_C\left((K_X + C)^{[r]}\right) \approx \mathcal{O}_C$.

Definition 18. [HK, 5.E] If $K_X + \Delta$ is an effective KLT pair in dimension 2, then for any ample divisor H', we can find $h \in \mathbb{R}_{>0}$ and an $\mathbb{R}$-divisor $H \sim_{\mathbb{R}} hH'$ such that $(X, \Delta + H)$ is klt and $K_X + \Delta + H$ is nef and big. Let

$$\lambda = inf\left\{t \geq 0 \mid K_X + \Delta + tH \ is \ nef\right\}.$$

If $\lambda = 0$, then $K_X + \Delta$ is nef, and thus (X, Δ) is minimal. If $\lambda > 0$, then, by Theorem 16, there exists a $(K_X + \Delta)$-negative extremal ray R such that $(K_X + \Delta + \lambda H).R = 0$. Then, c.f. [KK94, 2.3], if the corresponding contraction $\phi : X \to X'$ does not result in a log Del Pezzo surface or a birationally ruled surface, then setting $H' = \phi_* H$ and $\Delta' = \phi_* \Delta$, the divisor $K_{X'} + \Delta' + \delta H'$ is nef. Then the process may be repeated. The process either terminates at some step in a log minimal model or one of the aforementioned surfaces since there are no flips in dimension 2. The end result is a finite sequence of real numbers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ such that $K_{X_n} + \Delta_n + \lambda_n H_n$ is nef and $X \to X_n$ is a minimal model for $(X, \Delta + \lambda_n H)$.

The proofs in this article will also make use of several vanishing theorems.

Theorem 19. *(Kawamata-Viehweg Vanishing [Tan2, 2.11]) Let (X, Δ) be a projective klt surface (over an algebraically closed field of characteristic $p > 0$) where Δ is an effective $\mathbb{R}$-divisor. Let N be a nef $\mathbb{R}$-cartier $\mathbb{R}$-divisor with $\nu(X, N) \geq 1$. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D - (K_X - \Delta)$ is nef and big. Then there exists a positive real number $r(\Delta, D, N)$ such that $H^i(X, \mathcal{O}_X(D + rN + N')) = 0$ for every $i > 0$, every positive real number $r \geq r(\Delta, D, N)$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor N' such that $rN + N'$ is a Cartier divisor.*

Theorem 20. [Ek, II.6] *Let X be a minimal surface of general type and let $\mathcal{L}$ be an invertible sheaf that is numerically equivalent to $\omega_S^{\otimes i}$ for* *some* $i \geq 1$. *Then* $H^1(X, \mathcal{L}^{\otimes 2}) = 0$ *except possibly for certain surfaces in characteristic* 2 *with* $\chi(\mathcal{O}_X) \leq 1$ *.*

The following properties of base change will be used:

Theorem 21. [Har, III.10.2] *Let* $f : X \to Y$ *be a morphism of schemes of finite type over a field* k. *Then* f *is smooth of relative dimension* n *if and only if :*

(1) f *is flat; and*

(2) for each point $y \in Y$, *let* $X_{\overline{y}} = X_y \otimes_{k(y)} k(y)^-$ *where* $k(y)^-$ *is the algebraic closure of* $k(y)$. *Then* $X_{\overline{y}}$ *is equidimensional of dimension* n *and regular. (We say the "fibres of* f *are geometrically regular of equidimension* n.")

Theorem 22. [GW, 6.28] *Let* k *be a field,* X *a* k*-scheme locally of finite type, and let* $x \in X$ *be a closed point. Let* $d \geq 0$ *be an integer. We fix an algebraically closed extension* K *of* k, *and write* $X_K = X \otimes_k K$. *The following are equivalent:*

(i) The k*-scheme* X *is smooth of relative dimension* d *at* x.

(ii) For every point $\overline{x} \in X_K$ *lying over* x, X_K *is smooth of relative dimension* d *at* $\overline{x}$.

(iii) For every point $\overline{x} \in X_K$ *lying over* x, *the completed local ring* $\hat{\mathcal{O}}_{X_K,\overline{x}}$ *is isomorphic to a ring of formal power series* $K[[T_1, ..., T_d]]$ *over* K.

(iv) For every point $\overline{x} \in X_K$ *lying over* x, *the local ring* $\mathcal{O}_{X_K,\overline{x}}$ *is regular and has dimension* d.

(v) The equalities $dim_{\kappa(x)}T_x(X/k) = dim\mathcal{O}_{X,x} = d$ hold.

If these conditions are satisfied, then

(vi) The local ring $\mathcal{O}_{X,x}$ is regular and has dimension d.

Furthermore, if $\kappa(x) = k$, then the final condition implies the other ones.

As a consequence of the above theorems,

Proposition 23. *Let (X, Δ) be a terminal (resp. klt) pair which is simple normal crossings over a DVR R with perfect residue field k and perfect fraction field K. Let (X', Δ') denote the pair on the base change to a complete DVR R' with algebraically closed residue field. Then (X'_k, Δ'_k) and (X'_K, Δ'_K) are log-smooth and terminal (resp. klt.)*

Proof. By adjunction (X_k, Δ_k) and (X_K, Δ_K) are terminal (resp. klt) and snc by definition 10. Since smoothness is preserved by base-change, then all strata of (X', Δ') are smooth. Also, (X'_k, Δ'_k) is by definition log smooth after base change to the algebraic closure of k, since the algebraic closure of a perfect field is equal to its separable closure. Thus (X', Δ') is log-smooth.

Next, since k is assumed perfect, the notions of smooth and regular coincide. Thus by Theorems 22, and 21 we have that the base change $X \to X'$ over the algebraic closure k' is smooth of relative dimension 0, hence an étale base change. Thus by [Ko, 2.14,2.15], (X_k, Δ_k) and (X_K, Δ_K) are terminal (resp. klt) iff (X'_k, Δ'_k) and (X'_K, Δ'_K) are terminal (resp. klt). □

Definition 24. [Laz] Let D be a pseudoeffective $\mathbb{R}$-divisor on a normal projective variety X. The diminished base locus is defined as

$$\mathbb{B}_{-}(D) := \bigcup_{\substack{A \ \ Ample \\ D+A \ \ \mathbb{Q}-Cartter}} \mathbb{B}(D+A)$$

where $\mathbb{B}(D+A) = \bigcap_{n\geq 1} Bs(n(D+A))$ is the stable base locus.

Definition 25. Let D be a big Cartier divisor. Let F_m be the fixed divisor $|mD|_{fix}$. Then $F_{m+n} \leq F_m + F_n$ and the limit

$$N_{\sigma}(D) := \lim_{m\to\infty} \frac{1}{m} F_m$$

exists as an $\mathbb{R}$-divisor. We have $mN_{\sigma}(D) \leq F_m$ and for the $\mathbb{R}$-divisor $P_{\sigma}(D) := D - N_{\sigma}(D)$, we have an isomorphism

$$H^0(X, \mathcal{O}_X(mD)) \approx H^0(X, \mathcal{O}_X(\llcorner mP_{\sigma}(D)\lrcorner))$$

for any $m > 0$. The decomposition $D = P_{\sigma}(D) + N_{\sigma}(D)$ is called the sectional decomposition.

Remark 26. If D is not big, then $|mD|$ may be empty for certain positive integers m, and thus, in defining $N_{\sigma}(D)$, it is necessary to consider only the semigroup $\mathbb{N}(D)$ of m such that $|mD| \neq \emptyset$ for such D. In this article, only the sectional decomposition of big divisors is considered.

3. Proof of Main Result

The proof of the main result goes in several steps. The first step is to show that under more restrictive assumptions on the pair (X, Δ), that the invariance of plurigenera holds. The following is similar to [Tan3, 7.3] but without an ample boundary. The proof is similar to the Theorem [Suh, 1.2.1(ii)], where empty boundary is considered and there is, in addition, assumed a W_2 lifting hypothesis.

Proposition 27. *Let (X, Δ) be a terminal log smooth pair of relative dimension 2 over a DVR R with algebraically closed residue field k of characteristic $p > 2$ and perfect fraction field K. Assume that $K_X + \Delta$ is big and $\mathbb{Q}$-Cartier. Assume that $\mathbb{B}_- (K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$. Then there exists an m_0 such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0 (m (K_{X_k} + \Delta_k)) = h^0 (m (K_{X_K} + \Delta_K)).$$

The proof of proposition 27, is given in the following two claims.

Claim. Assumptions as above, after passing to an extension R' of R, there is a proper, smooth algebraic space X^{min}/R' and an R' morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef.

Proof. As k is algebraically closed of characteristic $p > 0$, then by the Cone Theorem 16,

$$\overline{NE} (X_k) = \overline{NE} (X_k)_{K_{X_k}+\Delta_k \geq 0} + \sum \mathbb{R}_{\geq 0} [C_i]$$

with each C_i is rational or $C_i = B_j$ for some B_j a component of Δ with $B_j^2 < 0$. Under the assumption that $\mathbb{B}_- (K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$, then actually each C_i is rational and is not a component of Δ_k. Thus $C_i.\Delta_k \geq 0$ so $C_i.K_{X_k} < 0$, and since $N_\sigma (K_{X_k} + \Delta_k)$ is a $\mathbb{Q}$-divisor by

[Tan, 7.1], then Theorem 17 implies $C_i \approx \mathbb{P}^1$. By Lemma 15, C_i can be contracted to a log-canonical point, which is actually a smooth point under the terminal assumption. Thus C_i is an exceptional curve of the first kind, so it is possible to apply Lemma 8.

By Lemma 8, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \to \tilde{X}$ induces the contraction of C_i on X_k and X_K, and further $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ is proper, smooth, separated, and finite type. Note that after a base change, the extension $\tilde{R} \supset R$ induces a finite extension on residue fields, and since k is algebraically closed, it induces identity on residue fields. Now I need to work with $\tilde{X}$. In order to apply the Cone Theorem 16 again, I need $\tilde{X}_k$ to be projective, but a smooth algebraic space of dimension 2, proper, separated, and of finite type over an algebraically closed field is projective, so $\tilde{X}_k$ is projective. Thus the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the generic fiber and an isomorphism on the special fiber, and blow-ups are preserved by flat base-change, so there is no problem in extending R. As the Picard number drops each step, there are only finitely many steps. □

Claim 28. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef.

Proof. This is a slight modification of [KU, 9.6]. By Lemma 8, it is impossible to reach a minimal X_K before a minimal X_k. Thus assume that $K_{X_k} + \Delta_k$ is nef but $K_{X_K} + \Delta_K$ is not nef. Let X^* denote the minimal model for X_K. Then letting $K_{X_K} + \Delta_K = f^*\left(K_{X^*} + \Delta_{X^*}\right) + E$, then, as (X_K, Δ_K) is terminal, $\left(K_{X^*} + \Delta_{X^*}\right)^2 > \left(K_{X_K} + \Delta_K\right)^2$.

For sufficiently large m such that $m\left(K_{X_k}+\Delta_k\right)$ is Cartier, then

$$h^0\left(m\left(K_{X_K}+\Delta_K\right)\right)=h^0\left(m\left(K_{X^*}+\Delta_{X^*}\right)\right).$$

Since $K_{X_k}+\Delta_{X_k}$ is nef and big, then for some $m\gg 0$ with $m\left(K_{X_k}+\Delta_{X_k}\right)$ also Cartier, then the higher cohomologies $H^i\left(m\left(K_{X_k}+\Delta_{X_k}\right)\right)$, $i>0$ vanish by Theorem 19. (The same thing happens on X_K by semi-continuity.) Thus, by the Riemann-Roch formula (see [Liu, Chapt. 9.1], [Har, III.7], [HS, A.4.2.1] and [BC, Cor. 9] for the necessary ingredients when K is perfect instead of algebraically closed) we have:

$$h^0\left(m\left(K_{X_K}+\Delta_K\right)\right)=h^0\left(m\left(K_{X^*}+\Delta_{X^*}\right)\right)$$
$$=\frac{1}{2}m^2\left(K_{X^*}+\Delta_{X^*}\right)^2+O\left(m\right).$$

Similarly:

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=\frac{1}{2}m^2\left(K_{X_k}+\Delta_{X_k}\right)^2+O\left(m\right).$$

So now there is a contradiction to semicontinuity in that, for some $m\gg 0$,

$$h^0\left(m\left(K_{X_K}+\Delta_K\right)\right)>h^0\left(m\left(K_{X_k}+\Delta_k\right)\right).$$

□

Now, by the above claim, there is a proper smooth algebraic space X^{min} such that X_k^{min} and X_K^{min} are obtained by blowing down -1 curves, and are minimal, proper, and smooth. Hence, by the Kawamata Viehweg vanishing (either Theorem 19, or the usual version in characteristic 0, depending on whether X has equal or mixed characteristic), there exists $m_0 \gg 0$ such that for $m > m_0$,

$$h^i\left(m\left(K_{X_K} + \Delta_K\right)\right) = h^i\left(m\left(K_{X_k} + \Delta_k\right)\right) = 0.$$

In fact, if $\Delta = 0$, then Ekedahl's vanishing Theorem 20 can be applied.

Thus, using the same Euler-characteristic argument as in the above claim, and using semicontinuity, it follows that for $m > m_0$,

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right).$$

The next step is to remove the restriction on the base locus from proposition 27.

Corollary 29. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 0$ and perfect fraction field K. Assume that $K_X + \Delta$ is big, $\mathbb{Q}$-Cartier, and simple normal crossings over R. Then, there exists an m_0 such that for $m \in m_0\mathbb{Z}^+$,*

$$h^0\left(m\left(K_{X_k} + \Delta_k\right)\right) = h^0\left(m\left(K_{X_K} + \Delta_K\right)\right)$$

Proof. Since the hypothesis and conclusion are preserved by base change, then extending R, we may assume that k is algebraically closed and that R is complete. The proof follows similarly to [HMX2010, 1.6] except that Proposition 27 is used in place of Kawamata Viehweg Vanishing. Recall that the log-canonical ring of $K_{X_k} + \Delta_k$ is finitely generated (c.f.

[Tan, 7.1]) so $N_\sigma(K_{X_k}+\Delta_k)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma(K_{X_k}+\Delta_k)$$

and let $0 \le \Theta \le \Delta$ be a $\mathbb{Q}$-divisor on X/R such that, by log-smoothness $\Theta|_{X_k} = \Theta_k$. Replacing (X,Δ) by a blow-up, assume (X,Δ) is terminal. By definition of N_σ, letting $m \gg 0$ and sufficiently divisible such that $m(K_{X_k}+\Theta_k)$ is integral, then

$$h^0(m(K_{X_k}+\Delta_k)) = h^0(m(K_{X_k}+\Theta_k))$$

and furthermore, by Proposition 27,

$$h^0(m(K_{X_k}+\Theta_k)) = h^0(m(K_{X_K}+\Theta_K)).$$

As $\Theta \le \Delta$, the theorem follows by semicontinuity. □

A result of Corollary 29 is the ability to run a minimal model program in mixed characteristic and relative dimension 2.

Proposition 30. *Let (X,Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k and perfect fraction field K. Assume that $K_X+\Delta$ is a big $\mathbb{Q}$-Cartier divisor, which is simple normal crossings over R. After possibly extending R, then the canonical ring Proj $R(K_X+\Delta)$ is finitely generated over R and a contraction of a $K_{X_k}+\Delta_k$-negative extremal ray cut out by an ample H on R induces a contraction of a $K_X+\Delta$ negative extremal ray. If the contraction is a flipping contraction, then the flip exists. Furthermore, a minimal* *model program with scaling can be run resulting in a minimal model for both geometric fibers.*

Proof. Assuming Corollary 29, suppose that there is a contraction of a curve on the closed fiber (X_k, Δ_k) corresponding to a $K_{X_k} + \Delta_k$ -negative extremal ray Σ_k. If necessary, extend R to be complete and k algebraically closed. Let $H_k = H|_{X_k}$ be such that $\left(K_{X_k} + \Delta_k + H_k\right).\Sigma_k = 0$. The goal is to show that

$$X_1 = Proj\ R\left(K_X + \Delta + H/R\right)$$

is finitely generated so that there exists a contraction defined by X_1.

Since by assumption, there is a contraction on X_k, for $m \gg 0$, $\bigoplus_{m\geq 0} H^0\left(ml\left(K_{X_k} + \Delta_k + H_k\right)\right)$ is generated in degree 1, so that there is a surjection $S^l H^0\left(m\left(K_{X_k} + \Delta_k + H_k\right)\right) \twoheadrightarrow H^0\left(ml\left(K_{X_k} + \Delta_k + H_k\right)\right)$. Consider the following diagram:

$$\begin{array}{ccc} S^l H^0\left(m\left(K_X + \Delta + H\right)/R\right) & \xrightarrow{\alpha} & H^0\left(ml\left(K_X + \Delta + H/R\right)\right) \\ \downarrow & & \downarrow \\ S^l H^0\left(m\left(K_{X_k} + \Delta_k + H_k\right)\right) & \xrightarrow{\twoheadrightarrow} & H^0\left(ml\left(K_{X_k} + \Delta_k + H_k\right)\right) \end{array}$$

(Note this argument implies the finite generation of $Proj\ R\left(K_X + \Delta\right)$ itself). The vertical maps are surjective by Corollary 29 and the bottom map is surjective by generation in degree 1. Then, since the generators of the bottom right module in the diagram over k lift to R by Nakayama's lemma, and henceα is surjective. This implies there is a contraction on R corresponding to Σ. By the same logic (Nakayama's lemma and the given surjectivity) if the divisorial contraction is a flipping contraction, then the flip exists, since it is defined by subtracting ϵH_k from $D = K_{X_k}+\Delta_k+H_k$ (as Σ_k is the only exceptional curve corresponding to $(K_{X_k} + \Delta_k + H_k) .\Sigma_k = 0$, then $-(K_{X_k} + \Delta_k + (1-\epsilon) H_k)$ is relatively ample.) Thus finite generation of $Proj\ R(K_{X_k} + \Delta_k + (1-\epsilon) H_k)$ implies by theorem 29 the finite generation of $Proj\ R(K_X + \Delta + (1-\epsilon) H)$.

Finally, let H a sufficiently general ample divisor on X and $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ be a sequence of real numbers corresponding to a minimal model program with scaling (c.f. definition 18) for $(X_k, \Delta_k + \lambda_n H_k)$, $H_k = H|_{X_k}$. By the proof of claim 3, at each step with $\lambda_i > 0$, $X \to X_i$ is a minimal model for $(X_k, \Delta_k + \lambda_i H_k)$ and $(X_K, \Delta_K + \lambda_i H_K)$ simultaneously. Thus, by claim 3, it is impossible to reach a minimal model on the generic fiber before having one on the special fiber. On the other hand, if a minimal model for $(X_k, \Delta_k + (\lambda_{n+1} = 0)\, H)$ is reached but $\lambda_{n+1} \neq 0$ on the generic fiber, by flat base extension, there is an extremal ray R on X such that $(K_{X_K} + \Delta_K + \lambda' H_K) .R = 0$ but $(K_{X_k} + \Delta_k + \lambda' H_k) > 0$ which is impossible by lemma 9. As the minimal model program with scaling does terminate for log surfaces we can thus simultaneously achieve a log minimal model on both fibers in this case of non-negative log kodaira dimensions. □

The following is another Corollary of Theorem 27, for pairs not of general type. This is a result of the proof of Theorem 27 not being affected by adding an ample divisor H. By abundance for log surfaces, this gives a log version of [KU].

Corollary 31. *Let (X, Δ) be a klt log-smooth pair of relative dimension 2 over a DVR R with residue field k and fraction field K such that k is algebraically closed of characteristic $p > 0$. Assume $K_X + \Delta$ is pseudo-effective. Then the numerical kodaira dimension satisfy*

$$\nu\left(K_{X_K} + \Delta_K\right) = \nu\left(K_{X_k} + \Delta_k\right).$$

As a result of abundance for log surfaces, the log kodaira dimension also satisfy

$$\kappa\left(K_{X_K} + \Delta_K\right) = \kappa\left(K_{X_k} + \Delta_k\right).$$

Proof. For any ample A, there exists an m_0 such that for $m > m_0$, $K_X + \Delta + \frac{1}{m}A = K_X + \Delta'$ is klt and and big (since $K_X + \Delta$ is assumed pseudo-effective). Thus the result follows from Corollary 29. □

4. Supplementary Algebra Results

This section will be removed from future versions.

4.1. **Riemann-Roch for Perfect Fields.** The standard Riemann-Roch theorem for algebraic surfaces is stated over an algebraically closed field. I'm sure the version for perfect fields is known if true, but since I was unable to find a reference, in this section I give necessary ingredients to extending the proof in [Har, Chapter 5.1] to perfect fields.

Lemma 32. [Liu, 7.3.16] *Let X be a projective variety over a field k. Let*

$$0 \to F \to G \to H \to 0$$

be an exact sequence of coherent sheaves on X. Then

$$\chi(G) = \chi(F) + \chi(H).$$

Theorem 33. [Liu, 7.3.17] *Let X be a projective curve over a field k. Let D be a Cartier divisor on X. Then we have*

$$\chi(\mathcal{O}_X(D)) = deg\ D + \chi(\mathcal{O}_X).$$

Definition 34. [Liu, 4.1.2, 8.3.1]A normal Noetherian integral domain of dimension 0 or 1 is called a Dedekind domain. A normal locally Noetherian scheme of dimension 0 or 1 is called a Dedekind scheme. Let S be a Dedekind Scheme. We call an integral, projective, flat S-scheme $\pi : X \to S$ of dimension 2 a fibered surface over S. If $dim\ S = 0$ then X is an integral, projective, algebraic surface over a field. An irreducible Weil divisor D is called horizontal if $dim\ S = 1$ and if $\pi|_D : D \to S$ is surjective. If $\pi(D)$ is reduced to a point, we say that D is vertical.

Theorem 35. [Liu, 9.1.12]*(Intersection on a fibered surface). Let $X \to S$ be a regular fibered surface. Let $s \in S$ be a closed point. Then there exists a unique bilinear map (of $\mathbb{Z}$-modules)*

$$i_s : Div(X) \times Div_s(X) \to \mathbb{Z}$$

which verifies the following properties:

(a) If $D \in Div(X)$ and $E \in Div_s(X)$ have no common component, then

$$i_s(D, E) = \sum_x i_x(D, E)\,[k(x) : k(s)],$$

where x runs through the closed points of X_s.

(b) The restriction of i_s to $Div_s(X) \times Div_s(X)$ is symmetric.

(c) $i_s(D, E) = i_s(D', E)$ if $D \sim D'$.

(d) If $0 < E \leq X_s$, then

$$i_s(D, E) = deg_{k(s)} \mathcal{O}_X(D)|_E.$$

Theorem 36. [Liu, 9.1.37] *Let $X \to S$ be a regular fibered surface, $s \in S$ a closed point, and $E \in Div_s(X)$ such that $0 < E < X_s$ (the second inequality is an empty condition if dim $S = 0$). Then we have*

$$\omega_{E/k(s)} \approx \left(\mathcal{O}_X(E) \otimes \omega_{X/S}\right)|_E,$$

and if $K_{X/S}$ is a canonical divisor,

$$p_a(E) = 1 + \frac{1}{2}\left(E^2 + K_{X/S} \cdot E\right).$$

Theorem 37. [Har, III.7.7] *Let X be a projective Cohen-Macaulay scheme of equidimension n over a field k. Then for any locally free sheaf $\mathscr{F}$ on X there are natural isomorphisms*

$$H^i(X, \mathscr{F}) \approx H^{n-i}(X, \mathscr{F}^\vee \otimes \omega_X^\circ)'$$

where ω_X° is the dualizing sheaf on X.

Theorem 38. [BC, Cor 9.] *Let X be a nonsingular projective variety of dimension n over a perfect field k. Then $\omega_X^\circ = \Omega_X^n$.*

Finally note that the following Riemann-Roch formula holds over perfect fields. The proof is the same as the one in [Har] Chapter 5, but the field is no longer assumed algebraically closed. All the necessary results are stated above.

Theorem 39. *Let D be a divisor on a nonsingular, projective surface X over a perfect field k. Then*

$$\chi\left(\mathcal{O}_X\left(D\right)\right) = \frac{1}{2}D.\left(D-K\right) + \chi\left(\mathcal{O}_X\right).$$

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