INVARIANCE OF PLURIGENERA IN POSITIVE CHARACTERISTIC ASSUMING LOG SMOOTH W2 LIFTINGS

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ABSTRACT. This is the end part of what was previously Surf-DIOPDP0.7 which is the Invariance of Plurigenerae in positive characteristic and non-negative kodaira dimensions assuming certain lifting conditions. Note that Suh has shown there is not always a lift of $K_X + \ulcorner \Delta \urcorner$ to $W_2(k)$- the point here is to assume such a lift exists (roughly equivalent to requiring that the Picard Scheme is reduced), and then to show that the plurigenera deform along a curve if certain other assumptions are met. (Needs a bit of fleshing out!)

1. Preliminaries

The discussion of deformations will mostly follow the book [Ser] where the theory of deformations, which is generally discussed for compact complex manifolds is extended to an arbitrary algebraically closed field. The underlying field k will therefore always be assumed to be algebraically closed.

1.1. **Deformations.**

1.1.1. *Derivations.*

Definition 1. [Abh] A derivation of a ring S with values in an S-module V is a homomorphism $D : S \to V$ of additive groups such that for all x, y in S, $D(xy) = xD(y) + yD(x)$. The set of all derivations of S with values in V is an S-module $Der(S, V)$. If $D(x) = 0$ for all x in a subring R of S, then we say that D is an R-derivation of S or a derivation of S/R. The submodule of all derivations of S/R with values in V is denoted by $Der_R(S, V)$. An R-derivation of S (or usually just an R-derivation) is an element of $Der_R(S, S)$. A derivation of S is an element of $Der(S, S)$.

Example 2. Let $R[x_1, ..., x_n]$ be a polynomial ring. Then ∂_{x_i}, the formal partial derivative with respect to x_i gives an R derivation.

Remark 3. Let R be a ring equipped with a derivation D. We extend D to the total ring of quotients Q of R as follows: If $h \in R$, then $D'(h) = D(h)$. If $h = \frac{u}{v}$ for u, v non-zero-divisors of R, then

$$D(u) = D'(u) = D'(vh) = vD'(h) + hD(v) = vD'(h) + \frac{u}{v}D(v).$$

Subtracting $\frac{u}{v}D'(v)$ from both sides and dividing by v gives

$$D'(h) = \frac{D(u)}{v} - \frac{u}{v^2}D(v) = \frac{vD(u) - uD(v)}{v^2}.$$

So D' is defined by the usual "quotient rule" from calculus existing in the total ring of quotients Q of a ring R. Now letting $u = \frac{u_1}{u_2}$ and $v = \frac{v_1}{v_2}$ be elements of Q such that u_i and v_i are non-zero divisors. Then $D'(u \cdot v) = D'\left(\frac{u_1 \cdot v_1}{u_2 \cdot v_2}\right) = uD'(v) + vD'(u)$ (after expanding) so in fact D' defines a derivation. Thus D' is the unique derivation on the total quotient ring which extends D.

Remark 4. Consider the total ring of quotients Q of a finitely generated k-algebra $S = k[X_1, X_2, ..., X_n] / (a_1, a_2, ..., a_m)$ with k a field. If f,g, and $f \circ g \in Q/S$ (invertible elements of the total quotient ring), then by the quotient rule (remark 3), we have

$$D(f(g(x)))' = D\left(\frac{f_1(g(x))}{f_2(g(x))}\right) = D[f(g(x))] \cdot D(g)$$

since expanding $f_i(g(x))$ and using the product and quotient rule gives a chain rule in $k[X_1, ..., X_n]$. If g is a vector of such elements, then

$$D\left(f\left(g^{(1)}(x), ..., g^{(n)}(x)\right)\right) = \sum_{j=1}^{n} \partial_{x^{(i)}} f\left(g^{(i)}\right) \cdot D\left(g^{(i)}(x)\right).$$

1.1.2. *Deformations.*

Definition 5. [Ser, 1.2.1]Let X be an algebraic scheme. A cartesian diagram of morphisms of schemes

$$\eta: \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ Spec(k) & \xrightarrow{s} & S \end{array}$$

where π is flat and surjective and S is connnected, is called a **deformation** of X over S. The deformation is called **first order** if $S = Spec(A)$ and $A = k[t]/(t)^2$ the ring of dual numbers. The deformation is called **infinitesimal** if $A \in Ob(\mathcal{A})$, where $\mathcal{A}$ is the category of local artinian k-algebras with residue field k. An infinitesimal deformation is called n^{th} order if $A = k[t]/(t^{n+1}) \in ob(\mathcal{A})$. A deformation of X over S is called **trivial** if it is isomorphic to the product family $X \times S \to S$. An infinitesimal deformation of X is called **locally trivial** if every point

$x \in X$ has an open neighborhood $U_x \in X$ such that

$$\begin{array}{ccc} U_x & \longrightarrow & \mathcal{X}_{|U_x} \\ \downarrow & & \downarrow \\ Spec\,(k) & \longrightarrow & S \end{array}$$

is a trivial deformation of U_x. All deformations of a nonsingular variety are locally trivial [Ser, 1.2.4].

1.2. **Kodaira Spencer Map.**

Theorem 6. [Ser, 1.2.9] *Let X be a nonsingular variety. Then there is a $1-1$ correspondence*

$$\kappa : \{ \textit{ isomorphism classes of first order deformations of } X\} \to H^1\,(X, T_X)$$

called the Kodaira-Spencer correspondence, where $T_X = Hom\,(\Omega^1_X, \mathcal{O}_X)$, such that $\kappa\,(\xi) = 0$ iff ξ is the trivial deformation. The correspondence κ is given as follows: Given a first order deformation

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec\,(k) & \longrightarrow & Spec\,(k\,[\epsilon]) \end{array}$$

choose an affine open cover $\mathcal{U} = \{U_i\}$ of X such that $\mathcal{X}|_{Ui}$ is trivial for all i. For each index i, we therefore have an isomorphism of deformations

$$\theta_i : U_i \times Spec\,(k\,[\epsilon]) \to \mathcal{X}|_{U_i}$$

and for each $i, j \in I$ *an automorphism*

$$\theta_{ij} := \theta_i^{-1}\theta_j : U_{ij} \times Spec\,(k\,[\epsilon]) \to U_{ij} \times Spec\,(k\,[\epsilon])\,.$$

$\theta = x + td\,(x)$ *for some* $k\,[\epsilon]$ *derivation, and the kodaira spencer map takes* θ *to* d*. The* $\{d_{ij}\}$ *forms a cech cocycle called the kodaira spencer class of the deformation.*

1.2.1. *Local Definition.* Now consider a first order deformation as above:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec\,(k) & \longrightarrow & Spec\,(k\,[\epsilon]) \end{array}$$

with an affine open cover $\mathcal{U} = \{U_i\}$ of X such that $\mathcal{X}|_{Ui}$ is trivial for all i. For each index i, we therefore have an isomorphism of deformations

$$\theta_i : U_i \times Spec\,(k\,[\epsilon]) \to \mathcal{X}|_{U_i}$$

and for each $i, j \in I$ an automorphism

$$\theta_{ij} := \theta_i^{-1}\theta_j : U_{ij} \times Spec\,(k\,[\epsilon]) \to U_{ij} \times Spec\,(k\,[\epsilon])\,.$$

For a given $U_i \cap U_j$, consider the element

$$\rho_{ij} = \sum_{\alpha} \frac{\partial {\theta_{ij}}^{(\alpha)}}{\partial t} \cdot \frac{\partial}{\partial z_i^{(\alpha)}}$$

where $\frac{\partial}{\partial t}$ denotes the formal derivative with respect to t. This defines a $k\,[\epsilon]$ derivation and, since $\theta_{ij} = \theta_i^{-1}\theta_j$ and on triple intersections

$\theta_{ij}\theta_{jk}\theta_{jk}^{-1} = 1_{U_{ijk}\times Spec(k[\epsilon])}$, then using the chain rule (remark 4), $\partial_t\theta_{ij} = \partial_t\left(\theta_{ik}\left(\theta_{kj}\left(z,t\right),t\right)\right) = \partial_t\theta_{kj} + \partial_t\theta_{ij}$ thus

$$\rho_{ij} - \rho_{ji} =$$

and

$$\rho_{ij} + \rho_{jk} - \rho_{ik} = 0$$

so that $\{\rho_{ij}\}$ is a Cech Cocycle giving an element of $H^1\left(X, T_X\right)$. By [Ser, 1.2.9], this cocycle uniquely defines the kodaira spencer class of the deformation.

1.3. **Obstruction Class.** (See [Lev, lemma 1.2]) Consider an infinitesimal deformation n^{th} order deformation defined over $D_{n+1} = Spec\left(k\left[t\right]/\left(t^{n+1}\right)\right)$:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec\left(k\right) & \longrightarrow & D_{n+1} \end{array} .$$

If $\mathcal{L}$ is an invertible sheaf on $X = X_0$, then a deformation of $\mathcal{L}$ over $\mathcal{X}$ is an invertible sheaf $\mathcal{L}'$ on $\mathcal{X}$ such that $\mathcal{L}' \otimes \mathcal{O}_{X_0} \approx \mathcal{L}$.

Suppose that a sheaf L on X_0 has an n^{th} order infinitesimal deformation to $X_n := X_0 \otimes D_n$. Consider the following short exact sequences:

$$0 \to k \overset{\times(t^n)}{\longrightarrow} D_n \overset{mod\ (t^n)}{\longrightarrow} D_{n-1} \to 0$$

and

$$0 \to D_{n-1} \overset{\times t}{\to} D_n \overset{mod\ t}{\longrightarrow} k \to 0.$$

As L is invertible, thus locally free, thus flat, then tensoring with L gives the following short exact sequences:

$$0 \to L_0 \to L_n \to L_{n-1} \to 0$$

and

$$0 \to L_{n-1} \to L_n \to L_0 \to 0.$$

Note the isomorphism $tL_n \approx L_{n-1}$ so these can be rewritten as

$$0 \to t^n L_n \to L_n \to L_{n-1} \to 0$$

and

$$0 \to tL_n \to L_n \to L_0 \to 0.$$

There is a map $f : t^n L_n \to tL_n$ given by the inclusion $k \to k[t]/(t^n)$ and another map $g : L_{n-1} \to L_0$ given by $k[t]/(t^n) \overset{mod\ t^n}{\longrightarrow} k$. Thus there is an induced diagram of long exact cohomology sequences (c.f. [Lev, equation 1]:

$$\begin{array}{ccccc} H^0(X_n, L_n) & \longrightarrow & H^0(X_n, L_{n-1}) & \xrightarrow{\partial} & H^1(X_n, t^n L_n) \\ {=}\downarrow & & \downarrow g & & \downarrow f \\ H^0(X_n, L_n) & \longrightarrow & H^0(X_n, L_0) & \xrightarrow[\partial]{} & H^1(X_n, tL_n) \end{array} .$$

Let $s \in H^0(X_n, L_0)$. Then $f(\partial(s)) = 0$ implies that $\partial(g(s)) = 0$. If we assume that s already lifts to n^{th} order, then $g(s) = \overline{s} \in H^0(X_n, L_0)$, and hence $\partial(\overline{s}) = 0$, so that $\overline{s}$ is in the image of the bottom left map. This implies $f(\partial(s))$ gives the obstruction to lifting $\overline{s}$ to $H^0(X_n, L_n)$.

1.3.1. *Local Definition.* This discussion follows [Lev, 1.2]. Suppose the situation of subsection 1.3 and assume s cuts out a Cartier divisor. Consider the map

$$\partial : H^0\left(X_n, L_{n-1}\right) \to H^1\left(X_n, t^n L_n\right).$$

By Cech cohomology (c.f. [Har]), this map is given by sending the global section $\{s_i\} \in H^0\left(X_n, L_{n+1}\right)$ to the cocycle defined by

$$\left\{t^{n+1}\zeta_{ij}\left(z_j\right)\right\} = \left\{s_i\left(g_{ij}\left(z_j,t\right),t\right) - \xi_{ij}\left(z_j,t\right)\cdot s_j\left(z_j,t\right) \ mod\ t^{n+2}\right\}.$$

Note that if $n = 0$, then in this case the maps f, g are identity, and hence (since it clearly extends to 0^{th} order)$\partial_1\left(s\right) = 0$. So

$$s_i\left(g_{ij}\left(z_j,t\right),t\right) - \xi_{ij}\left(z_j,t\right)\cdot s_j\left(z_j,t\right) \ \equiv 0\ mod\ \left(t^2\right)$$

and hence, writing $s_i\left(z_i,t\right) = \sum_{l=0}^{n} s_i^l\left(z_i\right)t^l$ as a polynomial in t, then the constant term of $\{t\zeta_{ij}\left(z_j\right)\}$ must be a coboundary. Thus applying the formal derivative operation $t\cdot\frac{\partial}{\partial t}$ to the above cocycle, we can simplify as follows:

$$s_i\left(g_{ij}\left(z_j,t\right),t\right) - \xi_{ij}\left(z_j,t\right)\cdot s_j\left(z_j,t\right) \ \equiv 0\ mod\ \left(t^2\right)$$

$$\left(n+1\right)t^{n+1}\zeta_{ij} \equiv$$

$$t\sum_{\alpha}\frac{\partial s_i}{\partial z^{(\alpha)}}\cdot\frac{\partial g_{ij}^{(\alpha)}}{\partial t} + t\frac{\partial s_i}{\partial t} - t\frac{\partial \xi_{ij}}{\partial t}\cdot s_j - t\xi_{ij}\cdot\frac{\partial s_j}{\partial t}\ mod\ t^{n+1}.$$

The term $t\left(\frac{\partial s_i}{\partial t} - \xi_{ij}\cdot\frac{\partial s_j}{\partial t}\right)$ defines a coboundary, so the above line is equivalent to

$$t\left(\underbrace{\sum_{\alpha}\frac{\partial s_i}{\partial z^{(\alpha)}}\cdot\frac{\partial g_{ij}^{(\alpha)}}{\partial t} - \frac{\partial \xi_{ij}}{\partial t}\cdot\frac{1}{\xi_{ij}}\cdot s_i}_{Ob(s)}\right)$$

so that under the isomorphism $L_n \xrightarrow{\times t} tL_{n+1}$ we must show the vanishing of the term $Ob\,(s)$ in parentheses.

Claim 7. On $U_i \cap U_j$, there is the identity $-\frac{\partial \xi_{ij}}{\partial t}\cdot\frac{1}{\xi_{ij}} = m\sum_{\alpha}\frac{\partial}{\partial z_i^{(\alpha)}}\left(\frac{\partial g_{ij}^{(\alpha)}}{\partial t}\right)$ for $L = mK_X$, Cartier.

Proof. This follows similarly to [Lev, 1.3], but I argue slightly differently. The transition functions ξ_{ij} for mK_X are given by $-m$ times the determinant of the matrix $A = \left(\frac{\partial g_{ij}^{(\alpha)}}{\partial z_j^{(\beta)}}\right)$. Note that as A is invertible on $U_i \cap U_j$, using the Jacobi identity, (mK_X Cartier implies we can use the chain rule c.f. remark 4)

$$-\partial_t det\,(A)^{-m}\cdot\frac{1}{det\,(A)^{-m}}$$

$$= m\cdot det\,(A)^{-m-1}\,\partial_t det\,(A)\cdot\frac{1}{det\,(A)^{-m}}$$

$$= m\cdot\frac{1}{det\,(A)}\cdot det\,(A)\,tr\left(A^{-1}\partial_t A\right).$$

Note that

$$\frac{\partial g_{ij}^{(\alpha)}}{\partial z_j^{(\beta)}}\cdot\frac{\partial z_j^{(\beta)}}{\partial z_i^{(\gamma)}} = \frac{\partial g_{ij}^{(\alpha)}}{\partial z_i^{(\gamma)}} = \frac{\partial z_i^{(\alpha)}}{\partial z_i^{(\gamma)}} = \delta_{\alpha\gamma}$$

and thus the inverse of A is given by $A^{-1}_{\beta\gamma} = \left(\frac{1}{k}\frac{\partial z_j^{(\beta)}}{\partial z_i^{(\gamma)}}\right)$, so the above becomes (with derivatives first evaluated with respect to t before substitutions are made):

$$= m\sum_{\alpha}\frac{\partial}{\partial t}\frac{\partial g_{ij}^{(\alpha)}}{\partial z_i^{(\alpha)}} = m\sum_{\alpha}\frac{\partial}{\partial z_i^{(\alpha)}}\frac{\partial g_{ij}^{(\alpha)}}{\partial t}.$$

□

By claim 7, $Ob\left(s\right)$ can be written as

$$\sum_{\alpha}\frac{\partial s_i}{\partial z^{(\alpha)}}\cdot\frac{\partial g_{ij}^{(\alpha)}}{\partial t} + m\sum_{\alpha}\frac{\partial}{\partial z_i^{(\alpha)}}\frac{\partial g_{ij}^{(\alpha)}}{\partial t}.$$

By [Eg, 2.7], if we add a simple normal crossings boundary Δ, written on the open set U_i as $\gamma_i = \prod_{\alpha\in I_i} z_{i,\alpha}^{q_\alpha}$, to L such that $m\left(K_X+\Delta\right)$, then $Ob\left(s\right)$ becomes

$$D\left(s\right) = m\sum_{\alpha}\frac{\partial}{\partial z_{i,\alpha}}\cdot\frac{\partial g_{ij,\alpha}}{\partial t}\cdot s_i$$

$$-\sum_{\alpha\in I_i}\frac{\partial g_{ij,\alpha}}{\partial t}\cdot\frac{mq_\alpha}{z_{i,\alpha}}\cdot s_i$$

$$+\sum_{\alpha}\frac{\partial g_{ij,\alpha}}{\partial t}\cdot\frac{\partial s_i}{\partial z_{i,\alpha}}$$

in $H^1\left(X_n, \mathcal{O}_{X_n}\left(m\left(K_{X_n}+\Delta_n\right)\right)\right)$.

2. Main Theorem

The idea of the proof is to use the arguments of [Lev] which rely only on degeneration of the Hodge-to-de Rham spectral sequence and not vanishing theorems. As this degeneration holds under certain conditions in characteristic p, and vanishing theorems are known to not necessarily hold, then this approach may be easier. Note that without the lifting assumption, there are actually counterexamples to invariance of plurigenera for elliptic surfaces (see [KU]). The following theorem of Deligne and Illusie is the key ingredient to the proof assuming a lifting actually exists:

Theorem 8. *([DI]) Let X be a proper smooth variety over a perfect field k of characteristic $p \geq dim\ X$ lifting to the ring $W_2(k)$ of the second Witt vectors. Then the hodge to de Rham spectral sequence:*

$$E_1^{ij} = H^j\left(X, \Omega^i\right) \implies H^*_{DR}(X/k)$$

degenerates in E_1.

Remark 9. The above theorem also has a relative version (c.f. [Ber]), so if we want to lift to n^{th} order, it is better to use the version where X/S, where S is a smooth k-scheme.

Proposition 10. *Suppose that X is a smooth proper variety over a perfect field of characteristic $p > k = dim(X)$. Assume that (X, Δ) is a klt, log smooth pair with a smooth map to an affine curve $\pi : X \to T = spec\ k[[t]]$. Let $S_n = Spec\ k[t]/(t^{n+1})$ and assume that for all n, $(X_n, \ulcorner\Delta_n\urcorner)$ has a lifting $\left(\tilde{X}_n, \ulcorner\tilde{\Delta}_n\urcorner\right)$ over $\tilde{S}_n = Spec\left(W_2\left(k[t]/(t^{n+1})\right)\right)$*

such that $\tilde{X}_n/\tilde{S}_n$ *is a log smooth lifting of* X_n/S_n. *Suppose*

$$H^0\left(m\left(K_{X_0}+\Delta_0\right)\right)$$

has a smooth section, and $m\left(K_X+\Delta\right)$ *is integral. Then the plurigenera*

$$h^0\left(m\left(K_{X_t}+\Delta_t\right)\right)$$

are independent of $t \in T$.

Proof. The proof is along the lines of [Lev] which the author previously generalized to the case of klt pairs in [Eg]. Cover X by coordinate neighborhoods U_i with coordinates (z_i). Let g_{ij} be transition functions for $\tilde{X}$ satisfying $z_i^{(\alpha)} = g_{ij}^{(\alpha)}$ on X . Let X_0 denote the special fiber and X_n be a scheme flat over $k\left[t\right]/\left(t^{n+1}\right)$ such that $X_n \times_{Spec\ k[t]/(t^{n+1})} k \approx X_0$. As in the characteristic 0 case (c.f. [Eg]), suppose s, a smooth section of $h^0\left(m\left(K_X+\Delta\right)/k\right)$, lifts to $H^0\left(K_{X_n}+\Delta_n\right)$ let $\overline{s}$ denote the image of the lift on the central fiber. As in [Eg], the obstruction to lifting $\overline{s} \in H^0\left(K_{X_0}+\Delta_0\right)$ to an element of $H^0\left(X_{n+1}, L_{n+1}\right)$ can be written as

$$Ob\left(s\right) = m\sum_{\alpha}\frac{\partial}{\partial z_{i,\alpha}}\cdot\frac{\partial g_{ij,\alpha}}{\partial t}\cdot s_i$$

$$-\sum_{\alpha\in I_i}\frac{\partial g_{ij,\alpha}}{\partial t}\cdot\frac{mq_\alpha}{z_{i,\alpha}}\cdot s_i$$

$$+\sum_{\alpha}\frac{\partial g_{ij,\alpha}}{\partial t}\cdot\frac{\partial s_i}{\partial z_{i,\alpha}}$$

in $H^1\left(X_n, \mathcal{O}_{X_n}\left(m\left(K_{X_n}+\Delta_n\right)\right)\right)$. This is defined through cech cohomology, and hence works in positive characteristic as well. Applying the same calculation as in ([Eg, 4.3]), $D(s) = d\mu$ where μ is a certain element of $H^1\left(Y_n, \Omega^{k-1}_{Y_n/T}\right)$, where Y_n is the m-fold cover $f : Y_n \to X_n$ branched over $m\left(K_{X_0} + \ulcorner \Delta_0 \urcorner\right)$. Also, using the same proof as [XieIII, 3.6], Y_n lifts to $W_2\left(k[t]/\left(t^{n+1}\right)\right)$ under the given lifting assumptions. The Hodge to de Rham spectral sequence on Y_n has E_1 page

$$\begin{array}{ccc}
H^0\left(\Omega^k_{Y_n}\right) & H^1\left(\Omega^k_{Y_n}\right) & \\
\uparrow & \uparrow & \\
H^0\left(\Omega^{k-1}_{Y_n}\right) & H^1\left(\Omega^{k-1}_{Y_n}\right) & \\
& & \\
\vdots & \vdots & \\
\uparrow & \uparrow & \\
H^0\left(\Omega^0_{Y_n}\right) & H^1\left(\Omega^0_{Y_n}\right) & \cdots
\end{array}$$

and degenerates in E_1(c.f. theorem 8 and remark 9), which means the map $d : H^1\left(\Omega^{n-1}_{Y_n}\right) \to H^1\left(\Omega^n_{Y_n}\right)$ is the zero map. Thus we can extend s to second order as $d\mu = D(s) = 0$. □

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