W2DIOP0.1

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ABSTRACT. This is the end part of what was previously Surf-DIOPDP0.7 which is the Invariance of Plurigenerae in positive characteristic and all kodaira dimensions assuming a lift of $(X, \ulcorner\Delta\urcorner)$. Note that Suh has show there is not always a lift of $K_X + \ulcorner\Delta\urcorner$ to $W_2(k)$- the point here is to assume such a lift exists (roughly equivalent to requiring that the Picard Scheme is reduced), and then to show that the plurigenera deform along a curve.

1. PRELIMINARIES

Definition 1. [Har, 9.13.1] Let X_0 be a scheme of finite type over a field k. Let $D = k[t]/t^{n+1}$. An n^{th} order infinitesimal deformation of X_0 is a scheme X', flat over D, and such that $X' \otimes_D k \approx X_0$.

2. MAIN THEOREM

The idea of the proof is to use the arguments of [Lev] which rely only on degeneration of the Hodge-to-de Rham spectral sequence and not vanishing theorems. As this degeneration holds under certain conditions in characteristic p, and vanishing theorems are known to not necessarily hold, then this approach may be easier. Note that without the lifting assumption, there are actually counterexamples to invariance of plurigenera for elliptic surfaces (see [KU]). The following theorem of

Deligne and Illusie is the key ingredient to the proof assuming a lifting actually exists:

Theorem 2. *([DI]) Let X be a proper smooth variety over a perfect field k of characteristic $p \geq dim\ X$ lifting to the ring $W_2(k)$ of the second Witt vectors. Then the hodge to de Rham spectral sequence:*

$$E_1^{ij} = H^j\left(X, \Omega^i\right) \implies H^*_{DR}(X/k)$$

degenerates in E_1.

Proposition 3. *Suppose that X is a smooth proper variety over a perfect field of characteristic $p > k = dim\,(X)$. Assume that (X, Δ) is a klt, log smooth pair with a smooth map to an affine curve $\pi : X \to T$ such that $(X_0, \ulcorner \Delta_0 \urcorner)$ admits a lifting to W_2, where X_0 is the fiber over $0 \in T$. Suppose $H^0(m(K_{X_0} + \Delta_0))$ has a smooth section, and $m(K_X + \Delta)$ is integral. Then the plurigenera*

$$h^0\left(m\left(K_{X_t} + \Delta_t\right)\right)$$

is independent of $t \in T$.

Proof. The proof is along the lines of [Lev] which the author previously generalized in [Eg]. Cover X by coordinate neighborhoods U_i with coordinates (z_i). Let g_{ij} be transition functions for $\tilde{X}$ satisfying $z_i^{(\alpha)} = g_{ij}^{(\alpha)}$ on X . Let X_0 denote the special fiber and X_n be a scheme flat over $k[t]/(t^{n+1})$ such that $X_n \times_{Spec\ k[t]/(t^{n+1})} k \approx X_0$. As in the characteristic 0 case (c.f. [Eg]), suppose s, a smooth section of $h^0(m(K_X + \Delta)/k)$, lifts to $H^0(K_{X_n}+\Delta_n)$ let $\overline{s}$ denote the image of the lift on the central fiber. As in [Eg], the obstruction to lifting $\overline{s}\in H^0(K_{X_0}+\Delta_0)$ to an element of $H^0(X_{n+1},L_{n+1})$ can be written as

$$D(s)=m\sum_{\alpha}\frac{\partial}{\partial z_{i,\alpha}}\cdot\frac{\partial g_{ij,\alpha}}{\partial t}\cdot s_i$$

$$-\sum_{\alpha\in I_i}\frac{\partial g_{ij,\alpha}}{\partial t}\cdot\frac{mq_\alpha}{z_{i,\alpha}}\cdot s_i$$

$$+\sum_{\alpha}\frac{\partial g_{ij,\alpha}}{\partial t}\cdot\frac{\partial s_i}{\partial z_{i,\alpha}}$$

in $H^1(X_n,\mathcal{O}_{X_n}(m(K_{X_n}+\Delta_n)))$. This is defined through cech cohomology, and hence works in positive characteristic as well. Applying the same calculation as in ([Eg], 4.3), $D(s)=d\mu$ where μ is a certain element of $H^1\left(Y_n,\Omega^{k-1}_{Y_n/T}\right)$, where Y_n is the m-fold cover $f:Y_n\to X_n$ branched over $m(K_{X_0}+\ulcorner\Delta_0\urcorner)$, . Also, by ([XieIII], 3.6), Y_0 lifts to W_2 under the assumption that $(X_0,\ulcorner\Delta_0\urcorner)$ lifts to W_2. The Hodge to de Rham spectral sequence on Y_0 has E_1 page

$$\begin{array}{ccc}
H^0\left(\Omega^k_{Y_0}\right) & H^1\left(\Omega^k_{Y_0}\right) & \\
\uparrow & \uparrow & \\
H^0\left(\Omega^{k-1}_{Y_0}\right) & H^1\left(\Omega^{k-1}_{Y_0}\right) & \\
\vdots & \vdots & \\
\uparrow & \uparrow & \\
H^0\left(\Omega^0_{Y_0}\right) & H^1\left(\Omega^0_{Y_0}\right) & \cdots
\end{array}$$

and degenerates in E_1, which means the map $d : H^1\left(\Omega_{Y_0}^{n-1}\right) \to H^1\left(\Omega_{Y_0}^{n}\right)$ is the zero map. Applying the induction to the exact sequence:

$$0 \to t\Omega_{Y_n}^{k-1} \to \Omega_{Y_n}^{k-1} \to \Omega_{Y_0}^{k-1} \to 0$$

and using the isomorphism $t\Omega_{Y_n}^{k-1} \approx \Omega_{Y_{n-1}}^{k-1}$ via the multiplication map, then the five lemma shows that $d : H^1\left(\Omega_{Y_n}^{k-1}\right) \to H^1\left(\Omega_{Y_n}^{k}\right)$ is also the zero map. Thus $d\mu = D(s) = 0$.

□

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