INVARIANCE OF PLURIGENERA FOR THREEFOLDS IN POSITIVE CHARACTERISTIC

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ABSTRACT. Log deformation invariance of plurigenera for a family of relative dimension 2 in assuming W_2 liftings of a pair, and also several generalizations of Suh's theorem on the Invariance of Plurigenera of general type threefolds without assuming the W_2 lifting assumption. Also a remark on the minimal model program in mixed characteristic.

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1. Theorem

The following is similar to ([Tan3], 7.3) but without an ample boundary, and without assuming the varieties in question are a priori minimal. The proof is similar to the theorem (1.2.1(ii) [Suh]), where empty boundary is considered, and there is a W_2 lifting hypothesis.

Proposition 1. *Let (X, Δ) be a terminal pair of relative dimension 2 over a DVR R with residue field k and fraction field K such that k is perfect of characteristic $p > 0$. Assume that $m(K_X + \Delta)$ is integral and $K_X + \Delta$ is big and $\mathbb{Q}$-Cartier. Assume (X_k, Δ_k) is terminal. Assume that $\mathbb{B}_-(K_{X_k} + \Delta_k) \wedge \Delta_k = \emptyset$. Then there exists an m_0 such that for $m \geq m_0$,*

$$dim_k\ H^0(m(K_{X_k} + \Delta_k)) = dim_k\ H^0(m(K_{X_K} + \Delta_K)).$$

Arguing as in the proof of ([HMX2010], 1.6) the above theorem has the following corollary

Corollary 2. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 5$ and fraction field K. Assume that $m(K_X + \Delta)$ is integral and $K_X + \Delta$ is big, log smooth over R, and $\mathbb{Q}$-Cartier. Assume (X_k, Δ_k) is klt. Then, there exists an m_0 such that for $m \geq m_0$,*

$$dim_k\ H^0(m(K_{X_k} + \Delta_k)) = dim_k\ H^0(m(K_{X_K} + \Delta_K))$$

A similar proof to proposition 1, gives the following:

Corollary 3. *Let* (X,Δ) *be a terminal pair of relative dimension* 2 *over a DVR* R *with residue field* k *and fraction field* K *such that* k *is perfect of characteristic* $p>0$. *Assume that* $mK_X+(m-1)\Delta$ *is integral and* $K_X+\Delta$ *is big and* $\mathbb{Q}$ *-Cartier. Assume* (X_k,Δ_k) *is terminal. Assume that* $\mathbb{B}_-\left(K_{X_k}+\frac{m-1}{m}\Delta_k\right)\wedge\Delta_k=\emptyset$. *Then there exists a computable* m_0 *such that for* $m\geq m_0$,

$$dim_k\ H^0\left(mK_{X_k}+(m-1)\Delta_k\right)=dim_k\ H^0\left(mK_{X_K}+(m-1)\Delta_K\right).$$

Arguing as in the proof of [HMX2010], yields the following:

Corollary 4. *Let* (X,Δ)*be a klt pair of relative dimension 2 over a DVR* R *with perfect residue field* k *of characteristic* p>5 *and fraction field* K. *Assume that* $m(K_X+\Delta)$ *is integral and* $K_X+\Delta$ *is big, log smooth over* R , *and* $\mathbb{Q}$ *-Cartier. Assume* (X_k,Δ_k) *is klt. Then, there exists a computable* m_0 *such that for* $m\geq m_0$,

$$dim_k\ H^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=dim_K\ H^0\left(m\left(K_{X_K}+\Delta_K\right)\right).$$

In fact, a simple argument on the pseudo-effective threshold shows that it suffices to assume $K_{X_k}+\Delta_k$ is big.

Corollary 5.

Another corollary of the above is the ability to run a minimal model program in mixed characteristic and relative dimension 2.

Corollary 6. *Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 5$ and fraction field K, and K is either of the same or mixed characteristic. Assume that $m(K_X + \Delta)$ is integral and $K_X + \Delta$ is big, log smooth over R , and $\mathbb{Q}$ -Cartier. Assume (X_k, Δ_k) is klt. Then a contraction of a $K_{X_k} + \Delta_k$-negative extremal ray induces a contraction of a $K_X + \Delta$ negative extremal ray.*

Without the boundary, and using a vanishing theorem of Ekedahl, there is also the following optimized version of the proposition 1.

Theorem 7. *Let X be a smooth general type variety of relative dimension 2 over a DVR R with residue field k and fraction field K such that k is algebraically closed of characteristic $p > 2$. Then*

$$dim_k\ H^0\left(mK_{X_k}\right) = dim_k\ H^0\left(mK_{X_K}\right)$$

for $m \geq 2$.

The following is another corollary of theorem 1, for pairs not of general type. This is a result of the proof of theorem 1 not being affected by adding an ample divisor H.

Corollary 8. *Let (X, Δ) be a klt log-smooth pair of relative dimension 2 over a DVR R with residue field k and fraction field K such that k is* *algebraically closed of characteristic* $p > 0$*. Then the numerical kodaira dimension satisfy* $\nu\left(K_{X_K} + \Delta_K\right) = \nu\left(K_{X_k} + \Delta_k\right)$*.*

By abundance for threefolds (c.f. [Wal]), this gives a log version of [KU], where the invariance of kodaira dimension is proved for non-log pairs.

Corollary 9. *Let* (X, Δ) *be a klt log-smooth pair of relative dimension* 2 *over a DVR* R *with residue field* k *and fraction field* K *such that* k *is algebraically closed of characteristic* $p > 0$*. Then the log kodaira dimension satisfy* $\kappa\left(K_{X_K} + \Delta_K\right) = \kappa\left(K_{X_k} + \Delta_k\right)$*.*

A consequence of corollary 8, using the proof of ([Eg]), I've sketched a proof of the following generalization of [Suh] to varieties not of general type. I'm actually not sure if it holds since without the W_2 lifting assumption, there are actually counterexamples to this statement c.f. [KU].

Theorem 10. *Suppose that* X *is a smooth proper threefold with a map to a over a perfect field of characteristic* $p > dim\left(X\right)$ *with a smooth morphism to a curve* T*. Assume that* (X, Δ) *is a klt log-smooth pair such that* $(X, \ulcorner\Delta\urcorner)$ *admits a lifting to* W_2*,* $h^0\left(m\left(K_{X_0} + \Delta_0\right)\right)$ *has a smooth section, and* $m\left(K_X + \Delta\right)$ *is integral. Then*

$$h^0\left(m\left(K_{X_t} + \Delta_t\right)\right)$$

is independent of $t \in T$*.*

2. Lemmas

Lemma 11. *([KU], 9.4) Let $\varphi : X \to Spec\,(R)$ be an algebraic space, proper, separated, and of finite type over $spec\,(R)$,where R is a DVR with residue field k_0 which is algebraically closed, and field of fractions k_1. Let X_1 (resp. X_0) denote the generic geometric (resp. closed) fibre of φ. If X_0 contains an exceptional curve of the first kind e, there exists a DVR $\tilde{R} \supset R$ and a proper smooth morphism $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \tilde{R} \to \tilde{X}$[1] over $Spec\left(\tilde{R}\right)$ such that on the closed fibre, π induces the contraction of the exceptional curve e. Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.*

Lemma 12. *([KU], 9.3) In the situation of the above lemma, if D, D' are divisors on X, then*

$$K_{X_0}^2 = K_{X_1}^2$$
$$(K_{X_0} \cdot D_0)_{X_0} = (K_{X_1} \cdot D_1)_{X_1}$$
$$(D_0 \cdot D_0')_{X_0} = (D_1 \cdot D_1')_{X_1}.$$

Lemma 13. *([KK94], 2.3.5) Let (S, B) be a log canonical surface over an algebraically closed field of characteristic $p > 0$. If $C \subset S$ is a curve with $C^2 < 0$ and $C.\,(K_S + B) < 0$, then $C \approx \mathbb{P}^1$ and it can be contracted to a log-canonical point.*

[1]This seems to be standard notation for deformations: $X \otimes \tilde{R} := X \times_{Spec\ R} \tilde{R}$

Lemma. *([Tan], 4.4) Let X be a projective normal surface and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-cartier. Let $\Delta = \sum b_i B_i$ be the prime decomposition. Let H be an $\mathbb{R}$-cartier ample $\mathbb{R}$-divisor. Then the following assertions hold:*

(1) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$

(2) $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i]$

(3) Each C_i in (1) and (2) is rational or $C_i = B_j$ for some B_j with $B_j^2 < 0$.

Theorem 14. *(Kawamata-Viehweg Vanishing , [Tan2] 2.11) Let (X, Δ) be a projective klt surface where Δ is an effective $\mathbb{R}$-divisor. Let N be a nef $\mathbb{R}$-cartier $\mathbb{R}$-divisor with $\nu(X, N) \geq 1$. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D - (K_X - \Delta)$ is nef and big. Then there exists a positive real number $r(\Delta, D, N)$ such that $H^i(X, \mathcal{O}_X(D + rN + N')) = 0$ for every $i > 0$, every positive real number $r \geq r(\Delta, D, N)$, and every nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor N' such that $rN + N'$ is a Cartier divisor.*

Theorem 15. *([Ek] II.1.6) Let X be a minimal surface of general type and let $\mathcal{L}$ be an invertible sheaf that is numerically equivalent to $\omega_S^{\otimes i}$ for some $i \geq 1$. Then $H^1(X, 2\mathcal{L}) = 0$ except possibly for certain surfaces in characteristic 2 with $\chi(\mathcal{O}_X) \leq 1$.*

Theorem 16. *(5.9, [DF]) Let X be a smooth surface in characteristic $p > 0$, and let D be a big and nef Cartier divisor D on X. Then*

$$H^1(X, \mathcal{O}_X(K_X + mD)) = 0$$

for all integers $m > m_0$*, where*

$m_0 = \frac{3}{p-1}$ *if* X *is quasi-elliptic with* $\kappa(X) = 1$*;*

$m_0 = \frac{2vol(X)+9}{p-1}$ *if* X *is of general type.*

Applying the proof of (9.1.18[Laz]), the above has the following corollary:

Corollary 17. *Let* (X, Δ) *be a smooth klt surface in characteristic* $p > 0$*, and let* D *be a big and nef Cartier divisor* D *on* X*. Then there exists a computable constant* $m > m_0$ *such that*

$$H^1(K_X + \Delta + mD) = 0.$$

3. Proof of Proposition 1

3.1. **Proposition 1 Proof.** Assume the hypothesis of theorem 1.

Claim. After passing to an extension R' of R, there is a proper, smooth algebraic space X^{min}/R' and an R' morphism $X \otimes R' \to X^{min}$ such that both $X_k \to X_k^{min}$ is obtained by successive blow-downs of (-1) curves and $K_{X_k^{min}} + \Delta_{X_k^{min}}$ is nef.

Proof. As X_k is algebraically closed of characteristic $p > 0$, then by the Cone Theorem 2,

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

with each C_i is rational or $C_i = B_j$ for some B_j a component of Δ with $B_j^2 < 0$. Under the assumption that $\mathbb{B}(K_X + \Delta) \wedge \Delta = \emptyset$, then actually each C_i is rational and is not a component of Δ. By lemma

13, C_i can be contracted to a log-canonical point (actually a smooth point under the terminal assumption). Thus C_i is an exceptional curve of the first kind, so it is possible to apply lemma 11.

By lemma 11, there is a DVR $\tilde{R} \supset R$ such that $\pi : X \otimes \tilde{R} \to \tilde{X}$ induces the contraction of C_i on X_k and X_K, and further $\tilde{\varphi} : \tilde{X} \to Spec\left(\tilde{R}\right)$ is proper, smooth, separated, and finite type. Now I need to work with $\tilde{X}$. In order to apply the cone theorem again, I need $\tilde{X}_k$ to be projective, but a smooth algebraic space of dimension 2, proper, separated, and of finite type over an algebraically closed field is projective, so $\tilde{X}_k$ is projective. Thus the same process can be repeated. Each extension $\tilde{R} \supset R$ induces a flat base change on the closed and generic fibers, and blow-ups are preserved by flat base-change, so there is no problem in extending R. As picard number drops each step, there are only finitely many steps. □

Claim 18. We also have $K_{X_K^{min}} + \Delta_{X_K^{min}}$ nef.

Proof. This is a slight modification of [KU], lemma 9.6. By lemma 11, it is impossible to reach a minimal X_K before a minimal X_k. Thus assume that $K_{X_k} + \Delta_k$ is nef but $K_{X_K} + \Delta_K$ is not nef. Let X^* denote the minimal model for X_K. Then letting $K_{X_K}+\Delta_K = f^*\left(K_{X^*} + \Delta_{X^*}\right)+E$, then, as (X_K, Δ_K) is terminal, $\left(K_{X^*} + \Delta_{X^*}\right)^2 > \left(K_{X_K} + \Delta_K\right)^2$.

For sufficiently large m such that $m\left(K_{X_k} + \Delta_k\right)$ is Cartier, then

$$h^0\left(m\left(K_{X_K} + \Delta_K\right)\right) = h^0\left(m\left(K_{X^*} + \Delta_{X^*}\right)\right).$$

Since $K_{X^*}+\Delta_{X^*}$ is nef and big, then for some $m \gg 0$ with $m\left(K_{X^*} + \Delta_{X^*}\right)$ also Cartier, then the higher cohomologies $H^i\left(m\left(K_{X^*} + \Delta_{X^*}\right)\right)$, $i > 0$

vanish by theorem 14. (The same thing happens on X_k which is already minimal.) Thus, by the Riemann-Roch formula for smooth surfaces over an algebraically closed field we have:

$$h^0\left(m\left(K_{X_K}+\Delta_K\right)\right)=\frac{1}{2}m^2\left(K_{X^*}+\Delta_{X^*}\right)^2+O\left(m\right)$$

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=\frac{1}{2}m^2\left(K_{X_k}+\Delta_{X_k}\right)^2+O\left(m\right).$$

So now there is a contradiction to semicontinuity in that, for some $m \gg 0$,

$$h^0\left(m\left(K_{X_K}+\Delta_K\right)\right)>h^0\left(m\left(K_{X_k}+\Delta_k\right)\right).$$

□

Now, by the above claim, there is a proper smooth algebraic space X^{min} such that X_k^{min} and X_K^{min} are obtained by blowing up -1 curves, and are minimal, proper, and smooth. Hence, by the Kawamata Viehweg vanishing (either theorem 14, or the usual version in characteristic 0, depending on whether X has equal or mixed characteristic), there exists $m_0 \gg 0$ such that for $m > m_0$,

$$h^i\left(m\left(K_{X_K}+\Delta_K\right)\right)=h^i\left(m\left(K_{X_k}+\Delta_k\right)\right)=0.$$

Now applying the Riemann-Roch formula and lemma 12 as in the above claim, we see that for $m > m_0$,

$$h^0\left(m\left(K_{X_k}+\Delta_k\right)\right)=h^0\left(m\left(K_{X_K}+\Delta_K\right)\right).$$

3.2. **Theorem 7 Proof.**

Proof. Running the minimal model program as in section 3.1, then again, after passing to an extension $R'$of R, there is a proper, smooth algebraic space $X^{min}/R'$and an R' morphism $X\otimes R'\to X^{min}$ such that both $K_{X_K^{min}}$ and $K_{X_k^{min}}$ are nef (X^{min}/R may not itself be minimal, but at least on both geometric fibers it is). Now applying Ekedahl's vanishing theorem 15, we have that $h^1(2K_{X_k})=h^1(2K_{X_K})=0$. By duality, $h^2(2K_{X_k})=h^2(2K_{X_K})=0$. Now by invariance of the Euler characteristic, we are done. □

3.3. **Corollary 3 Proof.** Apply Theorem 16 for vanishing instead of 14 in the proof of proposition 1.

3.4. **Corollary 2.** Proof. This follows with almost the same proof as ([HMX2010], 1.6) except that proposition [MainTheorem] is used in place of Kawamata Viehweg Vanishing (which doesn't necessarily hold for threefolds in characteristic p>0). Recall that the log-canonical ring is finitely generated so $N_\sigma(K_{X_k}+\Delta_k)$. Let

$$\Theta_k=\Delta_k-\Delta_k\wedge N_\sigma(K_{X_k}+\Delta_k)$$

and let $0\leq\Theta\leq\Delta$ be a $\mathbb{Q}$-divisor on X/R such that $\Theta|_{X_k}=\Theta_k$.By definition of N_σ,

$$dim_k\left|m(K_{X_k}+\Delta_k)\right|=dim_k\left|m(K_{X_k}+\Theta_k)\right|\qquad\star$$

and furthermore, by proposition [MainTheorem], letting m\gg0 and such that $m(K_{X_k}+\Theta_k)$ is integral, then

$$dim_k\left|m(K_{X_k}+\Theta_k)\right|=dim_K\left|m(K_{X_K}+\Theta_K)\right|.$$

As $\Theta \leq \Delta$, the theorem follows.

3.5. **Corollary 4.**

Claim 19. Let (X, Δ) be a klt pair of relative dimension 2 over a DVR R with perfect residue field k of characteristic $p > 5$ and fraction field K. Assume that $m(K_X + \Delta)$ is integral and $K_X + \Delta$ is big, log smooth over R, and $\mathbb{Q}$-Cartier. Assume (X_k, Δ_k) is klt. Then, there exists an m_0 such that for $m \geq m_0$,

$$dim_k\ H^0\left(m\left(K_{X_K} + \Delta_K\right)\right) = dim_k\ H^0\left(m\left(K_{X_k} + \Delta_k\right)\right).$$

Proof. This follows with almost the same proof as ([HMX2010], 1.6) except that proposition 1 is used in place of Kawamata Viehweg Vanishing (which doesn't necessarily hold for threefolds in characteristic $p > 0$). By finite generation of the log-canonical ring, $N_\sigma\left(K_{X_k} + \Delta_k\right)$ is a $\mathbb{Q}$-divisor. Let

$$\Theta_k = \Delta_k - \Delta_k \wedge N_\sigma\left(K_{X_k} + \Delta_k\right)$$

and let $0 \leq \Theta \leq \Delta$ be a $\mathbb{Q}$-divisor on X/R such that $\Theta|_{X_k} = \Theta_k$. By definition of N_σ,

$$dim_k\left|m\left(K_{X_k} + \Delta_k\right)\right| = dim_k\left|m\left(K_{X_k} + \Theta_k\right)\right| \qquad \star$$

Under the general type assumption

$$K_X + \frac{m}{m-1}\Delta \sim_{\mathbb{Q}} A + B$$

where A is ample and the support of X_0 is not contained in B. Let $0 < \delta \ll 1$ be such that $(X, \Delta + \delta(A+B))$ is terminal. Now

$$\underbrace{m(K_X+\Delta)}_{D} - \underbrace{(m-1-\delta)(\Delta-\Theta)}_{\Xi} \sim_{\mathbb{Q}}$$

$$K_X + \underbrace{\left(1 - \delta\frac{m}{m-1} + \delta\right)\Delta}_{\Phi} + \delta\left(K_X + \frac{m}{m-1}\Delta\right) + (m-1-\delta)(K_X+\Theta) \sim_{\mathbb{Q}}$$

$$K_X + \Phi + \delta(A+B) + (m-1-\delta)(K_X+\Theta) \sim_{\mathbb{Q}}$$

$$K_X + \Phi + \delta B + (m-1-\delta)(K_X+\Theta+H)$$

where $H = \frac{\delta}{m-1-\delta}A$.

As H is ample, $K_X + \Theta + H$ satisfies the hypothesis required to run a minimal model program simultaneously on both geometric fibers as in the proof of proposition 1. Let $f : X \to X^{min}$ be the map to a proper smooth algebraic space X^{min} such that X_k^{min} and X_K^{min} are obtained by blowing up -1 curves, and are minimal, proper, and smooth as in that proof. Then $f_K : X_K \to X_K^{min}$ and $f_k : X_k \to X_k^{min}$ give minimal models of both geometric fibers for $(X_K, \Theta_K + H_K)$ and $(X_k, \Theta_k + H_k)$. Let W resolve f in the following diagram such that p also resolves $(X, \Delta + A + B)$ on both geometric fibers.

$$\begin{array}{ccc} W & \xrightarrow{q} & X^{min} \\ {\scriptstyle p}\downarrow & \nearrow_{f} & \\ X & & \end{array}$$

As f is a birational contraction

$$f_*\left|m\left(K_{X_K}+\Theta_K\right)\right| \subset \left|mf_{K*}\left(K_{X_K}+\Theta_K\right)\right| = \left|mq_K^*f_{K*}\left(K_{X_K}+\Theta_K\right)\right| \qquad \star\star$$

Set $G = (m-1-\delta) f_* (K_X + \Theta + H)$ so that G is big and nef on both geometric fibers and

$$(m-1-\delta) p^* (K_X + \Theta + H) = q^* G + F$$

where $F \geq 0$ is exceptional for q.

Let W_k be the strict transform of X_k. Then $(X_k, \Phi_k + \delta B_k)$ is klt (as $\Phi_k < \Delta_k$ and $0 < \delta \ll 0$), and hence, since $p > 5$, strongly F-regular by ([Hara98]). Thus ([Das], 3.7) implies that

$$K_W + W_k = p^* (K_X + X_k + \Phi + \delta B) + E$$

where $\ulcorner E \urcorner|_{X_k} \geq 0$ is exceptional for p (in fact applying the adjunction is easier in this case, since everything is log smooth). Thus if

$$L = \ulcorner p^* (D - \Xi) + E - F \urcorner$$

, then

$$dim_k |L_K| \leq dim_k |D_K| \qquad \star\star\star .$$

Note that

$$L - W_0 \sim K_W + C + q^* G$$

with C the fractional part of $-p^* (D - \Xi) - E + F$. Thus applying theorem 17 to the euler characteristic argument used in the proof of proposition 1, we can find a computable m_0 such that for $m > m_0$, we have $dim_k |L_k| \leq dim_K |L|_{W_k}$.

By $\star, \star\star$, and $\star\star\star$, and semicontinuity, it remains to show that

$$dim_k \left|mq^* f_* \left(K_{X_k} + \Theta_k\right)\right| \leq dim_k \left|L_k\right|$$

This follows again as in ([HMX2010], 1.6).

$$\Xi = (m-1-\delta)(\Delta - \Theta) \leq m(\Delta - \Theta) \implies$$

$$D - \Xi \geq m(K_X + \Theta) \implies$$

$$\ulcorner L - mq^* f_* (K_X + \Theta) \urcorner \geq \ulcorner p^* (m(K_X + \Theta)) + E - F - mq^* f_* (K_X + \Theta) \urcorner.$$

H ample implies $p^* H \leq q^* f_* H$, hence

$$(m-1-\delta) p^* (K_X + \Theta + H) - (m-1-\delta) q^* f_* (K_X + \Theta + H) = F \implies$$

$$p^* (m(K_X + \Theta)) - q^* f_* (m(K_X + \Theta)) \geq F \implies$$

$$\ulcorner p^* (m(K_X + \Theta)) + E - F - mq^* f_* (K_X + \Theta) \urcorner \geq 0 \implies$$

$$\ulcorner L - mq^* f_* (K_X + \Theta) \urcorner \geq 0 \implies$$

$$L - \llcorner mq^* f_* (K_X + \Theta) \lrcorner \geq 0.$$

Since $m \gg 0$ was chosen such that $m(K_X + \Theta)$ is also integral, the proposition follows. □

3.6. **Corollary 6.** This section was suggested by my advisor (hopefully I've gotten the details right). Assuming corollary 2, suppose that there is a contraction of a curve on the closed fiber (X_k, Δ_k) corresponding to a negative extremal ray Σ_k. Let H_0 be such that $(K_{X_k} + \Delta_k + H_k).\Sigma_k = 0$. Let H be such that $H|_{X_k} = H_k$. The goal is to show that $X_1 = Proj\ \ (K_X + \Delta + H/R)$ is finitely generated so that there exists a contraction defined by X_1.

Since by assumption, there is a contraction on X_k, for $m \gg 0$, $\bigoplus_{k \geq 0} H^0 \left(ml \left(K_{X_k} + \Delta_k + H_k\right)\right)$ is generated in degree 1, so that there is a surjection

$$S^l H^0 \left(m \left(K_{X_k} + \Delta_k + H_k\right)\right) \twoheadrightarrow H^0 \left(ml \left(K_{X_k} + \Delta_k + H_k\right)\right).$$

Consider the following diagram:

$$\begin{array}{ccc} S^l H^0 \left(m \left(K_X + \Delta + H\right)/R\right) & \xrightarrow{\alpha} & H^0 \left(ml \left(K_X + \Delta + H/R\right)\right) \\ \downarrow & & \downarrow \\ S^l H^0 \left(m \left(K_{X_k} + \Delta_k + H_k\right)\right) & \xrightarrow{\twoheadrightarrow} & H^0 \left(ml \left(K_{X_k} + \Delta_k + H_k\right)\right) \end{array}.$$

The vertical maps are surjective by corollary 2, and the bottom map is surjective by generation in degree 1. By Nakayama's lemma, α is surjective, and thus there is a contraction on R corresponding to Σ.

4. Theorem 10 Proof

This section is a work in progress. The idea of the proof is to use the arguments of [Lev] which rely only on degeneration of the Hodge-to-de Rham spectral sequence and not vanishing theorems. As this degeneration holds under certain conditions in characteristic p, and vanishing theorems are known to not necessarily hold, then this approach may be easier. Note that without the lifting assumption, there are actually counterexamples to theorem 10 for elliptic surfaces (see [KU]). The following theorem of Deligne and Illusie is the key ingredient to the proof:

Theorem 20. *([DI]) Let X be a proper smooth variety over a perfect field k of characteristic $p \geq dim\ X$ lifting to the ring $W_2(k)$ of the second Witt vectors. Then the hodge to de Rham spectral sequence:*

$$E_1^{ij} = H^j\left(X, \Omega^i\right) \implies H^*_{DR}(X/k)$$

degenerates in E_1.

Proposition 21. *Suppose that X is a smooth proper variety over a perfect field of characteristic $p > k = dim\,(X)$. Assume that (X, Δ) is a klt pair such that $(X, \ulcorner\Delta\urcorner)$ admits a lifting to W_2, $h^0\,(m\,(K_X + \Delta)\,/k)$ has a smooth section, and $m\,(K_X + \Delta)$ is integral. Then*

$$h^0\,(m\,(K_X + \Delta)\,/K) = h^0\,(m\,(K_X + \Delta)\,/k)\,.$$

Proof. Let X_0 denote the special fiber and X_n be a scheme flat over $k\,[t]\,/\,(t^{n+1})$ such that $X_n \times_{Spec\ k[t]/(t^{n+1})} k \approx X_0$. As in the characteristic 0 case (c.f. [Eg]), suppose s, a smooth section of $h^0\,(m\,(K_X + \Delta)\,/k)$, lifts to $H^0\,(K_{X_n} + \Delta_n)$ let $\overline{s}$ denote the image of the lift on the central fiber. As in [Eg], the obstruction to lifting $\overline{s} \in H^0\,(K_{X_0} + \Delta_0)$ to an element of $H^0\,(X_{n+1}, L_{n+1})$ can be written as

$$D\,(s) = m \sum_{\alpha} \frac{\partial}{\partial z_{i,\alpha}} \cdot \frac{\partial g_{ij,\alpha}}{\partial t} \cdot s_i$$

$$- \sum_{\alpha \in I_i} \frac{\partial g_{ij,\alpha}}{\partial t} \cdot \frac{m q_\alpha}{z_{i,\alpha}} \cdot s_i$$

$$+\sum_{\alpha}\frac{\partial g_{ij,\alpha}}{\partial t}\cdot\frac{\partial s_i}{\partial z_{i,\alpha}}$$

in $H^1\left(X_n, \mathcal{O}_{X_n}\left(m\left(K_{X_n}+\Delta_n\right)\right)\right)$. This is defined through cech cohomology, and hence works in positive characteristic as well. Applying the same calculation as in ([Eg], 4.3), $D\left(s\right)=d\mu$ where μ is a certain element of $H^1\left(Y_n, \Omega^{k-1}_{Y_n/T}\right)$, where Y_n is the m-fold cover $f: Y_n\to X_n$ branched over $m\left(K_{X_0}+\ulcorner\Delta_0\urcorner\right)$, . Also, by ([XieIII], 3.6), Y_0 lifts to W_2 under the assumption that $\left(X_0, \ulcorner\Delta_0\urcorner\right)$ lifts to W_2. The Hodge to de Rham spectral sequence on Y_0 has E_1 page

$$\begin{array}{ccccc}
H^0\left(\Omega^k_{Y_0}\right) & & H^1\left(\Omega^k_{Y_0}\right) & & \\
\uparrow & & \uparrow & & \\
H^0\left(\Omega^{k-1}_{Y_0}\right) & & H^1\left(\Omega^{k-1}_{Y_0}\right) & & \\
 & & & & \\
\vdots & & \vdots & & \\
\uparrow & & \uparrow & & \\
H^0\left(\Omega^0_{Y_0}\right) & & H^1\left(\Omega^0_{Y_0}\right) & & \cdots
\end{array}$$

and degenerates in E_1, which means the map $d: H^1\left(\Omega^{n-1}_{Y_0}\right)\to H^1\left(\Omega^n_{Y_0}\right)$ is the zero map. Applying the induction to the exact sequence:

$$0\to t\Omega^{k-1}_{Y_n}\to\Omega^{k-1}_{Y_n}\to\Omega^{k-1}_{Y_0}\to 0$$

and using the isomorphism $t\Omega^{k-1}_{Y_n}\approx\Omega^{k-1}_{Y_{n-1}}$ via the multiplication map, then the five lemma shows that $d: H^1\left(\Omega^{k-1}_{Y_n}\right)\to H^1\left(\Omega^k_{Y_n}\right)$ is also the zero map. Thus $d\mu=D\left(s\right)=0$.

□

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